A continuum model for calculating snow avalanche velocities

HARALD NOREM
Norwegian Geotechnical Institute, PO Box 40 Taasen, N-0801 Oslo 8, Norway

FRIDTJOV IRGENS
Institute of Mechanics of the Technical University of Norway, N-7034 Trondheim-NTH, Norway

BONSAK SCHIELDROP
Industriell Hydro- og Aerodynamikk, Bogstadveien 11, N-0355 Oslo 3, Norway

ABSTRACT Constitutive equations for the properties of snow in avalanches are proposed. These are based on the equations for a modified Criminale - Ericksen - Filbey - fluid - which shows a close agreement to published experimental data on granular flow. The normal stresses are divided in three separate parts, effective pressure, pore pressure and a dispersive pressure. The shear stresses are caused by cohesion, Coulomb friction and dynamic stresses. The resulting terminal velocity is proportional to $h^{3/2}$ ($h$ = flow height). The validity of the model has an upper limit for the slope inclination, defined by the ratio of dynamic shear stresses and dispersive pressures, which is also in agreement with experimental results. The discussion of the assumptions shows that the model probably is reliable in the description of the flow, but the value of the material parameters need to be found experimentally.

MODELE CONTINU CALCULANT
LES VITESSES D'ECOULEMENT D'AVALANCHES
RESUME Des lois de comportement de la neige d'avalanche sont proposées. Ces lois sont basées sur les équations pour un fluide Criminale-Ericksen-Filbey, qui ont montré une bonne concordance avec des données expérimentales d'écoulement granulaire. Les contraintes normales sont divisées en trois parties, contrainte effective, pression interstitielle et une pression dispersive. Les contraintes de cisaillement sont causées par la cohésion, le frottement de Coulomb et des contraintes dynamiques. La vitesse terminale résultante est proportionnelle à $h^{3/2}$ (où h est la tête d'écoulement). La validité du modèle est limitée par une valeur maximum d'inclinaison de la pente, définie par le rapport des contraintes de cisaillement dynamiques et des contraintes dispersives. Cette observation est aussi confirmée par des résultats expérimentaux. Une examen des hypothèses de travail montre que le modèle est probablement fiable pour décrire l'écoulement étudié. Les valeurs des paramètres de la neige doivent cependant être déterminées expérimentalement.
INTRODUCTION

The Norwegian Geotechnical Institute (NGI) has since 1981 carried out full-scale experiments in the Ryggefonna avalanche path, Western Norway, recording avalanche velocities and impact pressures. The arrangements and preliminary results are reported by Norem et al (1985a), (1985b) and (1986). Evaluation of the results of these experiments clearly indicated a need for a better physical understanding of avalanche flow, especially in the lower part of the avalanche track and in the runout-zone. The aims of this project are:

- to give a physical description of the mechanics of the flow and the retarding mechanism.
- to identify the most important physical parameters that define the avalanche flow.
- to model the velocities and stresses in the lower part of the avalanche path and the runout distance with a reasonable accuracy.

However, the modelling of the flow in the run-out zone is still in progress.

EXISTING MODELS FOR ESTIMATING AVALANCHE VELOCITIES

Voellmy (1955) was the first to introduce a model for estimating avalanche velocities, and his model is still in extensive use. He assumed that the friction terms could be described by a dry friction, linearly increasing with the weight of the flowing snow layer and a dynamic drag proportional to the square of the avalanche velocity. These assumptions have also been used by Perla, Cheng and McClung (1980), who have refined the methods for estimating the velocities along the avalanche path.

It has been evident for many years that the Voellmy-model is too simple to model real avalanche velocities, and the highest inaccuracies are found in the lower part of the avalanche path. To obtain a better fit to the field measurements, Shaerer (1974) introduced a friction coefficient linearly dependent on the inverse of the avalanche velocity. Refinements have also been suggested by Lang et al (1985), who introduced "a fast-stop mechanism" in their viscosity parameter. Lang et al. (1979) had earlier introduced a friction parameter logarithmically dependent on the avalanche velocity.

The Voellmy as well as the Perla, Cheng, McClung-model, only takes into account the external forces acting on the avalanche mass as a whole, and cannot give any information on the distribution of stresses within this mass. Thus, internal velocity distributions cannot be obtained through the use of these models. A biviscous modified Bingham model was introduced by Dent and Lang (1983) to overcome this shortcoming, but the boundary conditions as well as the point of transition between the two fluids of different viscosity are difficult to define. The practical use of this model is thus limited.
BASIC LIMITATIONS AND ASSUMPTIONS

The fact that snow properties may vary between wide limits from one avalanche to another, and the fact that their volumes may differ by as much as several orders of magnitude, make it, of course, improbable that one model applicable to all types of avalanches can be constructed. Such a model would also have to take into account the variety of possible configurations of avalanche paths. Any practical model of avalanche motion must hence aim at only a limited part of the total field of avalanches and, probably, in very restricted path configurations.

This project concentrates on the dense part of the flow in the lower part of the avalanche path. Additionally, the path profile is considered to have no abrupt changes to avoid effects like hydraulic jumps and compressions of the flowing masses, and to have open slopes to satisfy the assumption of the avalanche as a two-dimensional problem.

CONSTITUTIVE EQUATIONS

Introduction

The literature of continuum mechanics provides a variety of constitutive equations for fluids having viscous, elastic and plastic properties. We assume that snow avalanches may be treated as a granular material where viscoplastic behaviour is predominant. In order to compare our snow model with other theoretical models and published experimental results, we shall choose constitutive equations which lead to simple forms for shear flows.

Steady simple shear flow

A steady simple shear flow may be defined by the velocity field

\[ v_x = \dot{\gamma} y, \quad v_y = v_z = 0 \quad (1) \]

x, y and z are cartesian coordinates. The shear rate \( \dot{\gamma} = dv_x/dy \) is a constant. In an isotropic material steady simple shear flow will introduce a shear stress \( \tau_{xy} \) and three normal extra stresses \( \tau_x, \tau_y \) and \( \tau_z \), cf. eq. (5). For an incompressible material the pressure \( p \), i.e. the flow-independent part of the normal stress, cannot be specified by a constitutive equation. Therefore, only normal stress differences can be measured and modelled. It is customary to introduce the following material functions:

Viscosity function \( \eta = \tau_{xy}/\dot{\gamma} \)

Primary normal stress coefficient \( \psi_1 = (\tau_x - \tau_y)/(\dot{\gamma})^2 \)  (2)

Secondary normal stress coefficient \( \psi_2 = (\tau_y - \tau_z)/(\dot{\gamma})^2 \)
It may be shown that the three viscometric functions $\eta$, $\Psi_1$, and $\Psi_2$ are even functions of the shear rate $\dot{\gamma}$.

Available material models

Several different constitutive equations have contributed to the equations representing our snow model. A purely dissipative material, without any elastic response, is defined by Goddard (1984) through the constitutive equations

$$\tau_{ij} = 2 \eta_{ijkl} D_{kl}$$

(3)

$\tau_{ij}$ is the extra stress tensor representing the flow-dependent stresses. $\eta_{ijkl}$ is a fourth-order viscosity tensor with 36 independent components depending on the deformation history. $D_{kl}$ is the rate of strain tensor.

$$D_{kl} = \frac{1}{2} \left[ \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right]$$

(4)

$v_k$ are the velocity components in a cartesian coordinate system with $x_k (k=1,2,3)$ denoting the three coordinates. In eq. (3) and in the equations to follow the standard Einstein summation convention is assumed. The viscosity tensor is positive definite with respect to the rate of strain tensor.

The total stresses $\sigma_{ij}$ in the material is a sum of an isotropic thermodynamic pressure $p$ and the extra stresses.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

(5)

$\delta_{ij}$ is a Kronecker delta and represents the unit tensor.

The material model represented by eq. (3) may describe thixotropy, non-linear viscosity and plasticity. A special version of eq. (3) represents the Oldroyd-Bingham material.

$$\tau_{ij} = 2 \mu_{ijkl} D_{kl}/\dot{\gamma} + 2\eta_{ijkl} D_{kl}$$

(6)

$\mu_{ijkl}$ is a fourth-order friction tensor depending on the deformation history. $\dot{\gamma}$ is the magnitude of the rate of strain tensor.

$$\dot{\gamma} = \sqrt{2D_{ij}D_{ij}}$$

(7)

In simple shear flow, this parameter is equal to the shear rate.

Another special version of (3) represents the Reiner-Rivlin fluid

$$\tau_{ij} = \alpha \delta_{ij} + \beta D_{ij} + \lambda D_{ik} D_{kj}$$

(8)

where $\alpha$, $\beta$ and $\lambda$ are functions of the principal invariants of $D_{ij}$.

In a steady simple shear flow eq. (8) yield
A continuum model for calculating snow avalanche velocities

\[ \tau_{xy} = \beta \dot{\gamma} / 2 \]

\[ \sigma_x = \sigma_y = -p + \alpha + \lambda(\dot{\gamma})^2 / 4, \quad \sigma_z = -p + \alpha \quad (9) \]

and \( \alpha, \beta \) and \( \lambda \) are now functions of the shear rate \( \dot{\gamma} \). In comparing eqs. (2) and (9) we see that the Reiner-Rivlin fluid has \( \Psi_1 = 0 \) and \( \Psi_2 = \lambda / 4 \). Experiments with non-Newtonian fluids have all shown that for materials exhibiting a non-zero value of \( \tau_y - \tau_z \) also show a non-zero value of \( \tau_x - \tau_y \). This experimental evidence has led to a rejection of this particular material model. We have not found any report on this matter regarding flow of granular materials. It may well be that the constitutive equations (8) is appropriate for describing the viscous response of a snow avalanche. However, we propose below a more general material model containing eq. (8) when \( \alpha = 0 \).

One of the most successful material models for stationary shear flow of non-Newtonian fluids appears to be the Criminale-Ericksen-Filbey-fluid (1958), the CEF-fluid,

\[ \tau_{ij} = 2 \eta D_{ij} + (2\Psi_1 + 4\Psi_2) D_{ik} D_{kj} \]

\[ -\Psi_1(D_{ij} - W_{ik} D_{kj} + D_{lik} W_{kj}) \quad (10) \]

\( \eta, \Psi_1 \) and \( \Psi_2 \) are the viscometric functions defined by eq. (2).

\( D_{ij} \) is the material derivative of the rate of strain tensor.

\( W_{ik} \) is the rate of rotation tensor.

\[ W_{ik} = \frac{1}{\nu} \left( \frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i} \right) \quad (11) \]

The constitutive equation (10) satisfies the principle of material frame indifference, Bird et al. (1977), as do the Reiner-Rivlin equation (8). In fact, eq. (10) reduces to (8) for \( \Psi_1 = \alpha = 0, 2 \eta = \beta \) and \( 4\Psi_2 = \lambda \).

A new material model for granular flow

We propose the following constitutive equations to represent a material model for granular materials in shear flow:

\[ \sigma_{ij} = -p\delta_{ij} + 2(a + bp_e^k) D_{ij} / \dot{\gamma} + 2\eta D_{ij} + (2\Psi_1 + 4\Psi_2) D_{ik} D_{kj} -\Psi_1(D_{ij} - W_{ik} D_{kj} + D_{lik} W_{kj}) \quad (12) \]

The effective pressure, \( p_e \), i.e. the part of \( p \) causing friction and the material parameters \( a, b \) and \( k \) will be discussed below. \( \eta, \Psi_1 \) and \( \Psi_2 \) are the viscometric functions of eqs. (2) and are thus functions only of the magnitude of the rate of strain tensor (7). The second term on the right hand side of eq. (12) is modelled according to the first term on the right hand side of eq. (6) and represents a plasticity response. The remainder of the right hand side of eq. (12) represents the viscoelastic response of the CEF-fluid.

The viscometric functions have to be modelled. For simplicity we propose power laws.
\[ \eta = \rho \ m \ (\dot{\gamma})^{n-1} \]  

(13)

\[ \Psi_1 = \rho \ \nu_1(\dot{\gamma})q^{-2}, \quad \Psi_2 = -\rho \nu_2(\dot{\gamma})r^{-2} \]  

(14)

where \( \rho \) is the mass density and \( m, n, \nu_1, q, \nu_2 \) and \( r \) are positive constant material parameters.

In a steady simple shear flow, as in eq. (1), the general constitutive equation (12) yields the following total stresses.

\[ \tau_{xy} = a + b \ p e^{k} + \rho \ m \ (\dot{\gamma})^n \]  

(15)

\[ \sigma_y = -p - \rho \ \nu_2(\dot{\gamma})r \]  

(16)

\[ \sigma_x = -p + \rho \ \nu_1(\dot{\gamma})q - \rho \nu_2(\dot{\gamma})r \]  

(17)

\[ \sigma_z = -p, \quad \tau_{yz} = \tau_{zx} = 0 \]  

(18)

For Newtonian and Bingham fluids the parameters will be

Newtonian: \( a = b = \nu_2 = \nu_1 = 0 \quad n = 1 \)

Bingham: \( b = \nu_2 = \nu_1 = 0 \quad n = 1 \)

Lang et al. (1985) proposed an avalanche model assuming:

\[ b = \nu_2 = \nu_1 = 0 \quad n = 2 \]

An admitted weakness in our model is that the influence of the rate of strain on the mass density, or the volume fraction of macroparticles, has not been specified. In addition the dependence on mass density and temperature of the material parameters need to be considered. Eqs. (15) - (18) may be used with variable mass density. However, in the applications of the constitutive equations (15) - (18) the selected parameters are assumed to be constant as long as the density and particle distribution are constant.

Discussion of the parameters

Equation (4.16), \( \sigma_y = -p - \rho \ \nu_2(\dot{\gamma})r \)

From a physical point of view the pressure part of the normal stresses is divided into two parts, \( p_e \), the effective pressure, i.e. the pressure transferred through the grain lattice, and \( p_u \), the pore pressure, i.e. the pressure of the interstitial fluid. In snow avalanches the pore pressure, \( p_u \), will be equal to the ambient pressure, \( p_1 \).

Eq. (4.15) may therefore be written

\[ \sigma_y = -p_e - p_u - \rho \nu_2(\dot{\gamma})r \]  

(19)

\( \rho \ \nu_2(\dot{\gamma})r \), is the dispersive pressure due to the collisions of particles moving relative to each other.
Equation (15), $T_{xy} = a + b p_e^k + p \mu(\dot{\gamma})^n$

The plasticity term, $a + b p_e^k$ in eq. (15) represents a shear strength and will be responsible for solidification of the snow, resulting in plug flow.

The cohesion parameter, $a$, is determined by the effect of sintering and by surface stresses between the snow macroparticles. The sintering process requires particle contact during a finite time interval, Gubler (1982), and will probably not be of importance in the presence of strain rates. The surface stresses are only present when the macroparticles are moist, Feda (1983), and they are linearly proportional to the relative contact area, Fukue (1979). The cohesion is thus dependent on the degree of saturation, volumetric density, particle size and particle distribution. We thus assume that the cohesion, $a$, will only have a non-zero value in snow avalanches with moist particle surfaces.

The friction parameter, $b$, has the same character as the static internal friction parameter of granular materials. According to Savage and Sayed (1984) the value of $b$ is very close to the tangent of the internal friction angle. The value of $b$ is dependent on the size, roughness and the moisture of the particles.

Stadler and Buggish (1985) have made viscosity experiments with granular materials at variable normal pressures and with different degrees of saturation, Fig. 1. As will be seen from Fig. 1, the shear strength increases linearly with the normal pressure. It is thus a reasonable assumption to set $k = 1$. The figure also shows that for dry particles, $s = 0\%$, the shear strength is insignificant at zero pressure. However, for moist particles, $s = 20\%$ and $s = 60\%$,
a certain cohesion will be present. Fig. 1 clearly indicates that the shear strength falls dramatically when the pores are filled up with an incompressible fluid, and that the shear strength is then less dependent on the pressure. Stadler and Buggish (1985) assume that for saturated materials, the increased pressure will cause a similar increase in the pressure of the interstitial fluid. Their explanation is also in accordance with the present theory in that the shear strength is dependent on the effective pressure.

Another experimental result that can be explained by the use of effective pressure to define the shear strength, is the viscometric experiments of Bagnold (1954). He used spheres of density equal to the density of the interstitial fluid, and his experiments showed, rather surprisingly, no evidence of shear strength. This is in accordance with the present theory as the effective pressure is zero for neutrally buoyant particles.

The theoretical investigations on the behaviour of the flow induced pressure, $\rho \nu_2(\dot{\gamma})^r$, and the dynamic shear stress, $\rho m(\dot{\gamma})^n$, were initiated by Bagnold (1954). His theoretical and experimental studies show that the ratio between $\rho m(\dot{\gamma})^n$ and $\rho \nu_2(\dot{\gamma})^r$ is constant for both variable shear rates and volumetric densities. The exponents $n$ and $r$ thus have equal values, and the ratio $m/\nu_2$ is constant for a given material. Bagnold (1954) found that $n=r=2$, when the dominant part of the energy loss was caused by macroparticle collisions. However, when the effects of fluid viscosity dominate, $n$ and $r$ were found to be 1. Bagnold's (1954) results have been confirmed in more recent experiments, e.g. Savage and Sayed (1984) found the values of $n$ and $r$ to be 2 at higher shear rates and at lower concentrations, and less than 2 at lower shear rates and at higher concentrations.

For the present model it is assumed that $n=r=2$ for the whole avalanche event. However, it would be more correct to assume that $n$ and $r$ will approach 1 as the avalanche decelerates to rest. To take this into account, however, will obviously complicate the calculations, and has for this reason not been considered here.

The material parameter, the ratio $m/\nu_2$, has been theoretically investigated by Savage (1984) and experimentally by Savage and Sayed (1984). Savage (1984) calculated the ratio to vary within 1.1 and 0.7 for $e$-values from 0 to 0.5. For snow avalanches the ratio is assumed to be as high as 0.8-1.0.

Savage and Sayed (1984) presented measurements of the ratio $m/\nu_2$ for various shear rates, based on viscometric experiments on dry materials. Their results for spherical polystyrene beads and crushed walnut shells are shown in Fig. 2.

The validity of the present model has been tested against the results of Savage and Sayed (1984). The expression for the ratio $|\tau_{xy}/\sigma_y+P_u|$ for cohesionless materials will be:

$$\frac{|\tau_{xy}|}{\sigma_y+P_u} = \frac{b}{P_e + \rho m(\dot{\gamma})^2} = \frac{b + m'(\dot{\gamma})^2}{1 + \nu_2'(\dot{\gamma})^2} \quad (20)$$

where, as can be seen,
A continuum model for calculating snow avalanche velocities

\[ \lim_{\gamma \to 0} \frac{\tau_{xy}}{\sigma_y} = b \]

\[ \lim_{\gamma \to \infty} \frac{\tau_{xy}}{\sigma_y} = \frac{m}{\nu_2} \]

Fig. 2 Experimental data of Savage and Sayed (1984) of the ratio shear to normal stresses for different shear rates. The curves represents the best fit of the present material equations.

The best fit of eq. (20) to the experimental results of Savage and Sayed (1984) is shown in Fig. 2. The results clearly indicate that the ratio at low shear rates is dominated by the quasi-static friction, and that dynamic effects dominate at higher shear rates. We take the correspondence of the present model with these experimental data as support of our model and in particular of the choice of \( k=1 \) and \( n=r=2 \).

STEADY SHEAR FLOW

The stresses \( \tau_{xy} \) and \( \sigma_y \) determined from the equations of motion

We consider a steady flow of snow down an inclined plane under the influence of gravity. The geometry and kinematics are presented in Fig. 3. The velocity field is given by \( v_x = v(y), \ v_y = v_z = 0 \).

The height of the snow layer is assumed to be constant and the boundary conditions are

\[ \sigma_y = -p_1 \text{ and } \tau_{xy} = 0 \text{ at } y = h \] (21)

The slip velocity, \( v(0) = v_0 \), has to be modelled separately.
The velocity field suggests that the extra stresses are functions of $y$ only. Furthermore, it is reasonable to assume that $\tau_{yz} = \tau_{zx} = 0$, as can be shown to be the case for an isotropic material. The equations of motion and the boundary conditions eq. (21) then yield

$$\tau_{xy} = \sin \phi \rho g dy = \rho g (h-y) \sin \phi$$  \hspace{1cm} (22)

$$\sigma_y = -p_1 \cos \phi \rho g dy = -p_1 - \rho g (h-y) \cos \phi$$  \hspace{1cm} (23)

where $\phi$ is the angle of inclination and $g$ the specific gravitational force.

Resulting flow velocity distributions

From eqs (15, 16, 22 and 23) we obtain an expression for the rate of shear, assuming $k = 1$, $n=r=2$:

$$\dot{\gamma} = \left[ \frac{g(\sin \phi - b \cos \phi)}{m - b v_2} \right]^{1/2} \left[ h - y - \frac{a}{\rho g(\sin \phi - b \cos \phi)} \right]^{1/2}$$  \hspace{1cm} (24)

Plug flow will be present for cohesive materials, $a>0$, at $\gamma(y) = 0$. The thickness of the plug flow will then be

$$h_p = \frac{a}{\rho g(\sin \phi - b \cos \phi)}$$  \hspace{1cm} (25)

Integrating eq. (24) and introducing the slip velocity gives the following expression for the avalanche terminal velocity profile:
A continuum model for calculating snow avalanche velocities

\[ v(y) = v_0 + \frac{2}{3} \left[ \frac{g(\sin \phi - b \cos \phi)}{m - b v_2^2} \right]^{\frac{1}{2}} (h - h_p)^{3/2} \left[ 1 - \left( \frac{v}{h - h_p} \right)^{3/2} \right] \]

for \( y \leq h - h_p \)

\[ v(y) = v_0 + \frac{2}{3} \left[ \frac{g(\sin \phi - b \cos \phi)}{m - b v_2^2} \right]^{\frac{1}{2}} (h - h_p)^{3/2} \]

for \( y > h - h_p \)

For cohesionless materials, \( a = 0 \), eq. (5.6) will be reduced to:

\[ v(y) = v_0 + \frac{2}{3} \left[ \frac{g(\sin \phi - b \cos \phi)}{m - b v_2^2} \right]^{\frac{1}{2}} h^{3/2} \left[ 1 - \left( \frac{v}{h} \right)^{3/2} \right] \]

The resulting velocity profile of eqs. (26, 27 and 28) is shown in Fig. 4, where it has been assumed that \( v_0 = 0.4 \, v_1 \), and \( v_1 = v(h) \).

![Fig. 4 Calculated velocity profiles.](image)

Discussion of the parameters

The boundary conditions at the bed surface

Some slip velocity at the bed surface is likely to exist for most avalanches, and the concept of slip velocity is inherent in most snow avalanche analyses, e.g. Voellmy (1955). The type of slip between the flowing material and the bed surface may be divided into three groups, real slip, apparent slip and erosion of a soft bed surface.
Lang et al. (1985) investigated several models, and probably the best of these is represented by:

$$\tau \big|_{y=0} = -b(\sigma_y + p_U) \big|_{y=0} + s v_0^2$$  \hspace{1cm} (29)

The friction parameter, $b$, is a snow to snow friction parameter and is considered to have the same value as $b$ in eq. (15). The constant $s$ is dependent on the material and roughness properties of the solid surface.

Combining eqs. (15, 16, 28 and 29) we obtain the velocities

$$v_0 = \left[ \frac{\rho g (\sin \phi - b \cos \phi)}{s} \right]^{1/2}$$

$$v_1 = \left[ \frac{\rho (m - b v_2)}{h^2 s} \right]^{1/2} + \left[ \frac{g (\sin \phi - b \cos \phi)}{m - b v_2} \right]^{1/2} h^{3/2}$$  \hspace{1cm} (30)

Eqs. (30) imply that the ratio $v_0/v_1$, is independent of the slope $\phi$. So far, no measurements of the slip velocities for snow avalanches are available.

The ratio $m/v_2$

Substitution of eqs. (16, 23 and 24) into eq. (15) gives the following expression for the effective pressure of a cohesionless granular material.

$$p_e = \frac{\rho g (h-y)}{m - b v_2} (m \cos \phi - v_2 \sin \phi)$$  \hspace{1cm} (31)

Eq. (31) predicts that the effective pressure increases linearly down to the bed surface, and decreases with increasing slope angle. The effective pressure is zero at $\tan \phi = m/v_2$. At this angle the dispersive pressure equals the overburden, and this angle thus constitutes an upper bound for the validity of our model. From a physical point of view this indicates that for angles exceeding this value air drag and energy losses at the top surface will contribute significantly. The effect of a critical angle in granular steady flow has been investigated theoretically by Savage (1984) and experimentally by Ishida and Hatano (1983).

Savage (1984) assumes that the normal pressure can be divided in a quasi-static and a dynamic part. This leads to an expression for the maximum angle of slopes where stationary flow can be obtained. The angle has been calculated to 25.6° for glass beads, and the value will vary with a factor of $\sqrt{1-e}$, $e$ is the coefficient of restitution.

Ishida and Hatano (1983) investigated granular flow in an aerated chute. With reduced supply of fluidizing air the density of the flow only slightly decreased with inclinations up to 24-26°. Above that inclination the flow had a turbulent-like character (splashing flow), and the density was considerably reduced as the inclination was increased further. Their experiments also indicated that without any supply of air, gliding flow existed up to 32°.

In our opinion the critical angle, $\phi = \tan^{-1} m/v_2$ equals the
transitional angle of Ishida and Hatano (1983) for the two kinds of flow patterns. Their transitional domain, 26°-32°, implies $m/v_2 = \tan \phi = 0.49 - 0.62$, which is in close agreement with the experimental results of Savage and Sayed (1984), who found $m/v_2 = 0.5-0.6$ for similar material.

At the critical angle the flow velocity for cohesionless materials will be:

$$v_1 | \tan \phi = m v_2 = \left[ \left[ \frac{\rho (m - bv_2)}{h^2 s} \right]^{\frac{1}{2}} + \frac{2}{3} \left[ \frac{g \sin \phi}{m} \right]^{\frac{1}{2}} h^{3/2} \right]$$

Equations (31 and 32) show that the effect of $b$ is reduced at higher angles and diminishes at the critical angle. As a preliminary assumption we propose to set $p_e = 0$ at angles above the critical.

Physically this means that the dispersive pressure is set equal to the overburden, and that only dynamic shear stresses are present above the critical angle.

**COMPARING THE PRESENT MODEL TO THE VOELLMY MODEL**

The Voellmy model predicts the terminal velocity:

$$v = (\xi h (\sin \phi - \mu \cos \phi))^{\frac{3}{2}}$$

$\xi$ is a roughness parameter and $\mu$ is a dry friction parameter.

The Voellmy-model estimates the same dependency between the slope angle as the present model for cohesionless materials. The roughness parameter, $\xi$, may also be expressed by the material parameters used for the present model

$$\xi = \left[ \frac{(\rho g)}{s} \right]^{\frac{1}{2}} + \left( \frac{4h^2}{9(m-bv_2)} \right)^{\frac{1}{2}}$$

Equation (34) implies that $\xi$ is strongly dependent on the flow height, which has also been suggested by several scientists after back-calculating real avalanches. This assumption has also been confirmed by recent radar experiments made by Gubler et al. (1986).

Fig. 5 shows $\xi$-values by Gubler et al. (1986) calculated related to the measured flow rate, $Q$. Fig. 5 also shows calculated $\xi$-values based on eq. (34) and for different assumptions for the magnitude of the slip velocity at $Q = 2000 \ m^3/s$. The figure indicates that the present model based on no slip at the bed surface has a good correlation to the presented results of Gubler et al. (1986).

Cohesive materials will have a different dependency between the terminal velocity and the slope angle as estimated by the Voellmy-model, since the thickness of the plug flow is dependent of the slope angle. It is thus not possible to make simple expressions for parametric studies with the Voellmy-model and the presented model for cohesive materials.
DISCUSSION

To simplify the mathematics the present model is based on several assumptions, of which the most important are:

Homogeneous particle size distribution

It was assumed by Bagnold (1954) that in granular flow the particles will separate, the smaller towards the bed surface and the greater towards the top of the flow. We are not aware of any measurements of particle size distributions in avalanches, but Bagnolds theory seems to have been confirmed for other kinds of granular flow, Davies (1966). According to Bagnolds theory, the viscosity parameter $m$ decreases towards the bed surface, thus making the velocity profile more convex than eq. (26) predicts.

No change of the material properties

The dissipation of energy within the avalanche mass will obviously raise the temperature of the snow and thereby, possibly, alter its material properties significantly. The potential energy in an avalanche with a vertical drop of 500 m is sufficient, theoretically, to raise the temperature of the snow 2.4°C. Although only a part of this total energy is dissipated to heat within the snow, changes of the snow properties due to temperature rise cannot in reality be excluded.

Constant volumetric density

Granular materials show an increase in the volumetric density when exposed to shear rates. The rate of volumetric increase is however,
most marked at lower shear rates. In addition, the volumetric density will also be affected by the effective stresses. The assumption of constant volumetric density is thus debatable.

Eq. (31) predicts an increase in the effective pressure with flow depth, and it may seems probable that the density may increase monotonically with increasing depth. It is possible to compensate for this effect by assuming a higher value than 1 for the exponent \( k \) in eq. (15) which will result in a more concave velocity profile. The effect of increasing density toward the bed surface is thus contrary to the previously assumed effect of separation.

The experiments of Stadler and Buggish (1985) have been made both at constant normal pressure and at constant density. At constant density the effect of shear rates has a higher proportionality coefficient than for experiments at constant normal pressure. This seems to indicate that there is some decrease of the density at higher shear rates when the effective pressure is decreasing. It is possible to compensate for this effect in the present model by assuming the exponents \( n \) and \( r \) somewhat less than 2.

Savage and Sayed (1984) make references to papers proving that the measured coefficient of restitution mostly will be reduced at higher impact velocities. Savage (1986) has calculated that this effect will reduce the values of \( n \) and \( r \).

It is thus reasonable to assume that \( n = r < 2 \). However, until more accurate investigations are published, the authors have selected \( n=r=2 \) for the whole avalanche event.

ACKNOWLEDGEMENT

This project is part of the Ryggfonn-project, and was initiated to establish a theoretical framework for the evaluation of the Ryggfonn-data.

The main project is mainly financed by the Norwegian Water Resources and Electricity Board, the Royal Norwegian Council for Scientific and Industrial Research (NTNF) and by the NGI. This sub-project is financed by the NTNF. The authors are grateful to the supporting agencies and will also express their gratitude to the NGI staff for fruitful discussions.

REFERENCES


Dent, J.D., Lang, T.E., (1983) A biviscous modified Bingham model
Gubler et al. (1986) Messungen an Fliesslawinen. Mitteilung no. 41 EISLF, Davos.

DISCUSSION

D.M. McClung

What does your model predict about the ratio of shear to normal forces at the boundary in the runout zone? In the example you showed about steady flow, it apparently predicts a low value at zero shear rate within the flow in agreement with the results of Savage and Sayed.
H. Norem
The minimum shear to normal stress ratio at the bed surface is assumed to be equal to the friction parameter, $b$, for cohesion-less materials.

R.L. Brown
Do the constitutive equations you used describe the dilatational effects associated with shearing flows? I'm asking this because as shearing increases, dilatational effects tend to change the material properties. This would become important in the runout zone.

H. Norem
The assumption of a constant density is questionable. However, as far as the effective pressure has a non-zero value, the variance of the density during the flow event is assumed to be small.

N. Maeno
1. You mentioned that the snow avalanche experiences a kind of phase change from solid to liquid as the shearing rate increases. I would like to ask if the constants $n$, $r$, etc. appearing in your constitutive equations have been determined as to incorporate the above concept.
2. I think your theory and calculation are based on an assumption of constant density. Does the calculated flow height correspond to the measured flow height of a natural avalanche of high and almost constant density?

H. Norem
1. The exponent $r$ and $n$ are dependent on the kind of energy loss in the granular flow. The constants are found to be 2 when the macroparticle collisions cause the dominant part of the energy loss, and $r$ and $n$ are 1 when fluid viscosity dominates.
2. We know that the flow height is very difficult to evaluate, since there probably is a continuous decrease of density from the dense flow to the snow cloud on top of the avalanche. However, the transitional zone is probably fairly narrow.

K. Hutter
Your cohesion coefficient is probably a constant. In that case I conjecture that you face difficulties to model runout zones. Your equations cannot reproduce hydrostatic conditions.

H. Norem
The cohesion coefficient, $a$, is a constant parameter in the model. We have assumed that, $a$, only has a non-zero value in snow avalanches with moist particle surfaces. Our model contains two parameters which may be responsible for a solidification of the flow: the cohesion coefficient and the friction parameter, $b$. In steady state flow at terminal velocity a plug will only be present if the cohesion parameter has a non-zero value. In the runout zone with decelerating flow both $a$ and $b$ are of importance. Our model does not apply for the plug region. We find, however, that this shortcoming can be circumvented if we, in addition to the boundary layer approximations, make the further assumption that the normal stresses are represented by the fluctuation pressure $p$. 