Some direct and inverse problems in land subsidence theory

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Abstract One of the effects of the underground human activities (geotechnological or civil engineering works) is the occurrence of land subsidence. The latter creates unfavourable conditions for the functioning of surface items (buildings, equipment, bands, etc.). In connection with this, the following two problems are considered and solved in this paper. (a) Direct problem — the mining system is given; determine the equation of the land subsidence. This problem is reduced to the Dirichlet problem for Fourier’s equation. (b) Inverse problem — the equation of the mining subsidence is given, determine the underground operations from which the given subsidence is realized. This problem is of extreme importance in the cases when mining has to be done under built-up areas. In these cases the building standards dictate the land subsidence equation. This necessitates a more rigorous formulation of a new inverse problem in the land subsidence theory. The problem is reduced to an inverse Dirichlet problem for Fourier’s equation. This is an incorrectly posed problem of mathematical physics in the sense of Hadamard. Its solution is obtained by Lions’ quasi-inversion method. Some generalizations of the posed problems are discussed. The posing and solving of the unique inverse problem in the applied geosciences allows to speak about laying the foundations of a modern land protection geo-engineering field.

INTRODUCTION

One of the effects of underground mining or civil engineering works is the occurrence of land subsidence. In connection with this phenomenon the following two problems arise (Fig. 1):

- **Prediction problem** (direct problem) — given the subsidence of the immediate roof, i.e. the mining system, determine the mining subsidence equation.
- **Protection problem** (inverse problem) — given the equation of the mining subsidence, dictated by the building norms for protection of the surface items, determine the underground operations which guarantee the required subsidence appearance.

Data assignment in the direct and indirect problems

In the direct problem the subsidence of the immediate roof \( w^o(x,0) = \varphi(x) \) (\( w^o \) = the vertical displacement of the rock mass points) is constructed either on the base of the
observations results or by the following procedure: above the mined area it is accepted that \( w^o = m\eta \), where \( m \) is the thickness of the seam, \( \eta \) is a coefficient which depends on the mining method (e.g. for back filling method \( \eta = 0.55 \pm 0.05 \)), and above the unmined area it is accepted that \( w^o = 0 \). The experience shows that this formulation of the "initial" conditions does work (Dimova, 1990; Litwiniszyn, 1974).

In the inverse problem the equation of the desired subsidence, as we noted already is constructed on the basis of the building norms for the surface sites preservation ("from...to..."). Consequently this equation is approximately given. Let us immediately emphasize that even if the equation were given exactly, errors would still exist. The reason is in the use of computers in solving these problems. Errors occur as a result of using numerical methods and rounding errors. The most important conclusion which follows here is that the subsidence equation is given approximately.

**PREDICTION PROBLEM**

The rock mass that is not influenced by openings can be separated from the rock mass that is influenced. Thereby the answer to the direct problem posed in the previous item (for convenience the plane problem is considered), gives the solution, for an appropriate scale, to the Dirichlet problem for Fourier's equation (Fig. 1):

\[
\dot{\zeta} = \int_0^z B(\lambda) d\lambda, \quad \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial \zeta}, \quad 0 \leq x \leq \pi, \quad 0 \leq \zeta \leq \zeta_H = \zeta_{z=H} \quad (1)
\]

\[
w^o(x,0) = \varphi(x), \quad 0 \leq x \leq \pi, \quad \zeta = 0 \quad (2)
\]

\[
w(0,\zeta) = w(\pi,\zeta) = 0, \quad 0 \leq \zeta \leq \zeta_H = \zeta_{z=H} \quad (3)
\]

where \( B = B(z) \) is a rock mass characteristic (Lattes & Lions, 1967; Dimova, 1990).

Problem (1)-(3) is correctly posed, because it satisfies all Hadamard’s requirements (Tikhonov & Arsenin, 1977; Dimova, 1990b):

- existence of a solution,
- uniqueness of the solution,
- stability of the solution, i.e. small changes in the initial data should lead to small changes in the solution.

When solving the prediction problem we are not faced with any difficulties.
PROTECTION PROBLEM

When mining operations take place under built-up areas, the problem, as we mentioned above, is inverse (Dimov & Dimova, 1994b; 1987; Dimova, 1990b): given the subsidence equation \( w(x,\xi_H) = \psi(x) \), determine the subsidence of the immediate roof (i.e. sequence of mining) \( w(x,0) = \varphi(x) \).

This problem can be formulated as follows:

\[
\zeta = \int_0^z B(\lambda) d\lambda, \quad \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial \xi}, \quad 0 \leq x \leq \pi, \quad 0 \leq \xi \leq \xi_H \tag{4}
\]

\[
w(x,\xi_H) = \psi(x), \quad 0 \leq x \leq \pi, \quad \xi = \xi_H \tag{5}
\]

\[
w(0,\xi) = w(\pi,\xi) = 0, \quad 0 \leq \xi \leq \xi_H \tag{6}
\]

where the function \( \psi = \psi(x) \) is initially given and the function searched is \( w = w(x,0) \).

Now we will prove that problem (4)-(6) is incorrect in Hadamard’s sense and therefore requires a careful treatment.

Let us suppose that function \( w(x,\xi) \) is a solution to problem (4)-(6) and let us introduce function \( w(x,0) = \chi(x), \quad 0 \leq x \leq \pi \). It is clear that function \( w(x,\xi) \) can be considered as a solution to Dirichlet’s problem for equation (4) with boundary conditions:

\[
w(x,0) = \chi(x), \quad w(0,\xi) = w(\pi,\xi) = 0 \tag{7}
\]

and consequently (Genchev, 1988) the function \( \psi(x) = w(x,\xi_H) \) proves to be limitless times differentiable. If this property is not available, problem (4)-(6) is unsolvable. It is not difficult to prove that this problem is incorrect even in the class of the infinitely smooth functions. In order to make sure that this is true, it is sufficient to add the following function to one solution of problem (4)-(6):

\[
\nu_n(x,\xi) = \frac{\varepsilon}{n^k} \exp\{-n^2(\xi-\xi_H)\} \sin nx \tag{8}
\]

which satisfies both equation (4) and conditions \( \nu(0,\xi) = \nu(\pi,\xi) = 0 \). Although we obviously have:

\[
|\frac{\partial^\nu}{\partial x^\nu} \nu_n(x,\xi_H)| \leq \varepsilon \quad \text{for} \quad \nu = 0, 1, \ldots, k \tag{9}
\]

the sequence \( \{\nu_n(x,0), n = 1,2,\ldots\} \) is not limited. In this way the statement is proved. This shows that we cannot "restore" the subsidence of the immediate roof (the initial conditions; the cause for the phenomenon) provided we know the earth’s subsidence equation (the result). It follows directly from the fact that equation (4) describes also the phenomenon "heat conductivity" (Fourier), i.e. an irreversible process and is not invariant with respect to the change \( \tau = -\xi \). Generally speaking, John’s problem for the heat conductivity equation with "back time" is an incorrect problem (in (4) \( \xi \) is a "timelike" variable) (John, 1955).
There are a number of methods for solving incorrect problems (Tikhonov & Arsenin, 1977; Dimova, 1990). Here we will discuss the method proposed by Lions (Lattes & Lions, 1967; Tikhonov & Arsenin, 1977; Dimova, 1990). According to this method, instead of a concrete incorrect problem, a correct one similar to the incorrect one (in the sense of defined measure for similarity $\pi$) is considered. Thus in place of equation (4), the enhanced equation is considered (Babenko, 1986):

$$\frac{\partial w}{\partial \xi} = \frac{\partial^2 w}{\partial x^2} + \epsilon \frac{\partial^4 w}{\partial x^4}, \quad \epsilon > 0, \quad 0 \leq x \leq \pi, \quad 0 \leq \xi \leq \xi_H \quad (10)$$

with boundary and initial conditions:

$$w(x,\xi_H) = \psi(x), \quad 0 \leq x \leq \pi, \quad \xi = \xi_H \quad (11)$$

$$w(0,\xi) = \frac{\partial^2 w(0,\xi)}{\partial x^2} = 0, \quad w(\pi,\xi) = \frac{\partial^2 w(\pi,\xi)}{\partial x^2} = 0 \quad (12)$$

Problem (10)-(12) is correctly posed and its solution has the form:

$$w(x,\xi) = \sum_{k=1}^{+\infty} a_k \exp\{\lambda_k \xi\} \sin kx \quad (13)$$

where $\lambda_k = -k^2 + \epsilon k^4$.

If $\psi(x) = \sum_{k=1}^{+\infty} a_k \sin kx$, then

$$w(x,\xi) = \sum_{k=1}^{+\infty} a_k \exp\{\lambda_k (\xi - \xi_H)\} \sin kx \quad (14)$$

the function:

$$w(x,0) = \sum_{k=1}^{+\infty} a_k \exp\{-\lambda_k \xi_H\} \sin kx \quad (15)$$

is the sought approximate solution of the incorrect problem (4)-(6) (Babenko, 1986). If we now solve problem (1)-(3) with the "initial" data (15), we will obtain:

$$\varphi(x,\xi_H) = \sum_{k=1}^{+\infty} a_k \exp\{-\epsilon k^2 \xi_H\} \sin kx \quad (16)$$

a function which can be considered as an element of some compact $Y$, similar to the initial function $\psi(x)$.

We propose that for a determined measure for similarity $\eta$, $\epsilon$ should be determined from the equation (Babenko, 1986):

$$\sum_{k=1}^{+\infty} a_k^2 (1 - \exp\{-\epsilon k^4 \xi_H\})^2 = \pi^2 \sum_{k=1}^{+\infty} a_k^2 \quad (17)$$

We have applied Lions's method many times successfully. The results are encouraging.
SOME GENERALIZATIONS

Let us point out, that here we considered two basic problems in a bounded domain:
- prediction problem, also called direct problem,
- protection problem, also called inverse problem.

Now let us draw our attention to the following: If we are based on the integro-geometrical theory (influence function) for mining subsidence (Whittaker & Reddish, 1989), then the solution to the prediction problem, i.e. the problem for determining function \( w(x, \xi_H) = \psi(x) \) is reduced to the solution of the integral:

\[
\int_{a}^{b} K(x - \xi) \phi(x), \quad c < x < \pi
\]

for a given function \( \phi(\xi) \) and \( K(x - \xi) \) - influence function, which according to different authors (King, Bals, Knothe, etc.) and for different regions, takes different forms (Whittaker & Reddish, 1989); \( a \) and \( b \) are the boundaries of the mined-out area, \( c \) and \( d \) are the boundaries within which the solution is searched.

The protection problem consists in seeking a solution to Fredholm's integral equation of the first kind (18), i.e. for a given function \( K(x - \xi) \) and \( \phi(x) \), let us determine the function \( w(x,0) = \phi(\xi) \). This problem has an instable solution, hence it is incorrect in J. Hadamard's sense (Kolmogoroff & Fomin, 1981). When searching for the solution to equation (18), we apply Tikhonov's regularization method (Tikhonov & Arsenin, 1977; Lavrentiev et al., 1980). We considered both the 3-D case, and the case with an inaccurately given kernel \( K(x - \xi) \) (Dimova, 1994b and Dimova & Dimov, 1994a). Let us stress here that the kernel is constructed on the basis of the results from the survey measurements. This is what causes the errors in setting \( K(x - \xi) \).

If we consider the treated problems in the light of the functional analysis, we can say that the protection problem is reduced to seeking a solution to the operator equation:

\[
A\phi = \psi, \quad \phi \in \Phi, \quad \psi \in \Psi
\]

where \( \Phi \) and \( \Psi \) are some functional spaces and \( A \) is a completely continuous linear operator (the operator is such in (18) (Kolmogoroff & Fomin, 1981; Lavrentiev et al., 1980)) which acts from \( \Phi \) to \( \Psi \). It is known that this problem is incorrectly posed, because operator \( A^{-1} \), inverse to operator \( A \), exists and is not continuous (Lavrentiev et al., 1980; Tikhonov & Arsenin, 1977; Dimova, 1995). It is also known that if spaces \( \Phi \) and \( \Psi \) are properly selected, problem (19) can be transformed into a correctly posed problem (Lavrentiev et al., 1980; Kalitkin, 1978). However this method is not constructive because of the inaccurate setting of the right part in (19) (Lavrentiev et al., 1980; Kalitkin, 1978).

CONCLUSION

Finally let us summarize:
- The protection problem is incorrectly posed in Hadamard's sense and in solving it, special methods have to be used. The classical mathematical methods do not work here (Tikhonov & Arsenin, 1977; Lattes & Lions, 1967; Dimova, 1990).
The primitive "selection method", applied in mining practice, i.e. the inverse problem solved by numerous direct problems and comparing the result with the "desired" result, has a limited sphere of influence here. It can be applied only for the protection of one or two objects in the case of the plane problem. For more objects and the 3-D problem, the application of the "selection method" is impossible.

In summary we should note that we were the first to formulate the original protection problem, to introduce the terms "inverse problem" and "protection problem" and to show the way of solving it in 1985 (Dimov & Dimova, 1985). Our claims have been confirmed by an independent source (Review Journal, 1988). The monograph (Dimova, 1990) devoted to the inverse problems in the mine subsidence theory is unique in the world mining literature and reflects our results until 1990. The new results obtained after 1990 are described in the other references given here (Dimov & Dimova, 1994a; Dimova & Dimov, 1994b; Dimova, 1994a,b, 1993).

Our studies lay the foundations of the modern theory of protection of surface sites in mine areas.

Note: The theory presented here can be applied in studying subsidence arising from groundwater withdrawal, oil and gas field activities and underground coal gasification (see also (Whittaker & Reddish, 1989)). About some other applications of the subsidence theory see (Dimova, 1995).

REFERENCES


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