A general formulation for saturated aquifer deformation under dynamic and viscous conditions

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Abstract A general formulation is presented for dynamic and viscous deformation of saturated sedimentary material. Based on first principles (such as conservation of momentum and mass), constitutive laws (for both pore fluid and the solid matrix) and the effective stress principle, a complete mathematical description is developed for aquifer movement. The governing equation derived in this paper not only generalizes the dynamic governing equations derived by Fukuo (1970) and Zienkiewicz & Bettess (1982), but can be further reduced to various quasi-dynamic equations (consolidation or subsidence equations) including those of Terzaghi (1925), Biot (1941), Mikasa (1965) and Helm (1987). The new formulation unifies multiple physical processes such as wave, diffusion and creep into a single powerful governing equation with surprisingly few parameters. Self-contradictions of physical assumptions introduced by Fukuo and Zienkiewicz & Bettess are discussed.

INTRODUCTION
The dynamic behaviour of saturated porous material interests investigators in many fields such as earthquake engineering and geotechnical engineering. Biot (1962) investigated acoustic propagation through deformable and saturated porous media. Zienkiewicz & Bettess (1982) generalized Biot’s formulation and gave a more general mathematical description that is widely applied to problems investigated by earthquake and geotechnical engineers. Fukuo (1970) in a paper presented in the first international symposium on land subsidence in Tokyo formulated the governing equation for movement of saturated soil under dynamic conditions. In all these previous investigations, the significance of relative acceleration $a_r$ between the solid and fluid phases and the physical constraints required for correctly applying Darcy-Gersevanov’s law are ignored. As a result, some self-contradictions become intermixed in the theory. In this paper, a new formulation is developed.

FUNDAMENTAL RELATIONS

Mass conservation

Assuming that no mass is destroyed or created within a control volume of interest in space and that mass conservation holds for each individual phase (water or soil phase)
gives the following equations:

\[ \frac{\partial \rho_s}{\partial t} + \nabla (\rho_s v^s) = 0 \]  

\[ \frac{\partial \rho_w}{\partial t} + \nabla (\rho_w v^w) = 0 \]  

where \( \frac{\partial}{\partial t} \) is the partial derivative with respect to time, superscripts "s" and "w" denote soil and water, \( \rho_s \) and \( \rho_w \) are water and soil densities respectively, \( v^s \) and \( v^w \) denote phase velocities (bold means vector or tensor everywhere in this paper). If one multiplies each term in (1) by \( (1 - n) \) and in (2) by \( n \) and then sum, by noting the definition of the bulk flux \( q^b (= n v^w + (1 - n) v^s) \) introduced by Helm (1987), one has an alternative mass conservation expression for bulk flow (Helm, 1987):

\[ \nabla \cdot q^b = \frac{1 - n}{\rho_s} \frac{d \rho_s}{dt} - \frac{n}{\rho_w} \frac{d \rho_w}{dt} \]  

where \( n \) is porosity, \( d/dt \) denotes the material derivative with respect to time \( t \). When the density of each phase is constant (or all constituents are incompressible), the right-hand side of (3) equals zero, namely \( \nabla q^b = 0 \) which represents an incompressibility condition.

**Momentum conservation**

**Momentum equation for a two phase mixture**

Momentum conservation for saturated porous material (namely, soil and water, a two phase mixture) is given (Li, 1994) by:

\[ \nabla a^s + \nabla a^w + \rho^b b = \rho^s a^s + \rho^w a^w \]  

where bulk density \( \rho^b \) can be expressed as \( \rho^b = (1 - n) \rho_s + n \rho_w \) and equals the sum of the phase densities \( \rho^s (= (1 - n) \rho_s) \) and \( \rho^w (= n \rho_w) \) that are weighted by the volume ratio occupied by each phase, \( a^s \) is the acceleration of the skeletal matrix, \( a^w \) of interstitial water, and \( a^w \) is a second order tensor that represents the total stress on water defined by:

\[ \nabla \sigma^w = n \nabla (p \delta) - n \nabla \tau^w \]  

\( \delta \) (or \( \delta_{ij} \)) is the Kronecker delta, \( p \) is water pressure, \( \tau^w \) represents the viscous stress on the cutting surface of the element of interest including through water (fluid-fluid viscous stress) and the water-soil interface (fluid-solid viscous stress). A total stress \( \sigma^w \) gradient on the solid skeleton is given by:

\[ \nabla \sigma^w = \nabla a^s + (1 - n) \nabla p \delta + n \nabla \tau^s \]  

where \( \sigma^s \) is effective stress (Terzaghi, 1925), and \( \tau^s \) denotes stress on the solid-water interface.

In this paper, it is assumed for simplicity that the component of fluid-fluid viscous stress is negligibly small when compared to the fluid-solid viscous stress component (namely, assuming \( \tau^s = \tau^w = \tau \)).
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Following the solution for viscous force on the surface of a single sphere (Landau & Lifshitz, 1975) and introducing the concept of average drag force per unit volume $F_{\text{drag}} (= \nabla \tau)$ on a control element of interest in space (Li, 1994) give:

$$F_{\text{drag}} = \left( \frac{\rho_w}{2e} \right) m^{-1} a' + \left[ 3 \rho_w \left( \frac{ng}{2eK\pi} \right)^{0.5} \right] m^{-1} \int_{-\infty}^{t} \frac{dv'}{\sqrt{t-\tau}} d\tau + \gamma_w K^{-1} q \quad (7)$$

where $e$ is the void ratio and $m (=K/\bar{K}$ where $K$ and $\bar{K}$ are hydraulic conductivity in forms of a matrix and a scalar) is a second order directional tensor that accounts for anisotropic material (Li, 1994). For isotropic material, $m$ reduces to $\delta$. In (7), laminar Newtonian flow of water relative to the skeleton is assumed. The term $a' (=a^w - \bar{a}^s$) is relative acceleration, $v'^r (=v^w - \bar{v}^s$) is relative velocity, $q (=nv')$ is specific discharge, $g$ is gravitational acceleration, $\gamma_w$ is the unit weight of water.

**Governing equation for solid matrix movement**

Based on momentum conservation, a governing equation of aquifer movement is now developed. If momentum is conserved individually for each phase (see (4)), one can reach an expression derived by Li (1994):

$$\left[1 + \frac{1}{eG} \right] a^s - \nabla \sigma - \nabla \tau - \left[1 - \frac{1}{G} \right] b - \frac{a^b}{nG} = 0 \quad (8)$$

where $G (=\rho_s/\rho_w)$ is the specific weight of the solid phase, $a^b (=n a^w + (1 - n) \bar{a}^s$) is defined as bulk acceleration.

Introducing (7) into (8) by eliminating $\nabla \tau(=F_{\text{drag}})$ gives the governing equation for the movement of the skeletal matrix, namely:

$$c_1 a^s - c_2 v'^s - c_3 \nabla a' - c_4 b + c_5 \dot{q}^b + c_6 = 0 \quad (9)$$

where $\dot{q}^b$ is the time derivative of $q^b$ and coefficients $c_1$ through $c_6$ are given by:

$$c_1 = \delta (1 + /eG) - m^{-1}/2n^2G \quad (10)$$

$$c_2 = \left\{ \frac{d(lnm)}{dt} \right\} \left( \frac{\delta G n - m^{-1}/2n^2G}{\delta \sigma} + \gamma_w K^{-1}/\rho^s \right) \quad (11)$$

$$c_3 = \frac{1}{\rho^s} \quad (12)$$

$$c_4 = \frac{n}{G - 1} \quad (13)$$

$$c_5 = \frac{m^{-1}}{2n^2G} - \frac{\delta}{nG} \quad (14)$$

$$c_6 = 3\rho_w \sqrt{ng/2eK\pi} \frac{m^{-1} f(t)}{\rho^s} \quad (15)$$
One of the advantages of (9) is that if \( \theta^p \) is written in explicit terms, one can conveniently apply the incompressibility condition. Equation (9) is a new governing equation and describes solid matrix movement under dynamic and viscous conditions.

**General relation of drag and driving forces for viscous laminar flow** By combining momentum conservation for the fluid only and an average drag force on the solid matrix (7), one obtains a general relation between drag and driving forces (Li, 1994) by:

\[
K^{-1} f^d = -J
\]

where the right-hand term \( J \) denotes a dimensionless driving force and the left-hand product is a dimensionless drag force. \( f^d \) and \( J \) are given by:

\[
f^d = \{ \frac{K}{2 \epsilon g} a^r + \left[ n \frac{K/c_\epsilon \gamma^{0.5}}{2} \right] \left\{ \int_{-\infty}^{t} \frac{d\gamma}{\sqrt{t-\tau}} \right\} + q \} = \{ \}
\]

\[
J = \left[ \frac{\partial \gamma^w}{\partial t} + (v^p \cdot \nabla) \gamma^w - (v^p \cdot \nabla) \right] /g + (\nabla p \delta) / \gamma^w + \nabla z(t)
\]

where \( z(t) \) is the elevation of an arbitrarily selected point of interest (usually fixed in the solid phase) whose velocity is \( v^p \). When the first and second terms in (17) are negligibly small when compared to the third term \( q \) due to small relative acceleration \( a^r \), drag force \( f^d \) defined by (17) reduces to specific discharge \( q \). Correspondingly, when the first two terms in (18) are negligibly small (for example, if \( a^w/g \) is small when compared to the last two terms in (18)), then the driving force defined by (18) can be reduced to the gradient of hydraulic head. Accordingly, equation (18) simplifies to Darcy-Gersevanov’s law and eventually to Darcy’s law when \( v^w \gg v^s \) (Li, 1994). It is important to emphasize that equation (16) is a general relation of drag and driving force which is derived from momentum conservation of flowing pore fluid. Darcy-Gersevanov’s law and Darcy’s law are simply special cases of (16).

**Constitutive law for skeletal matrix** Following Helm’s (1992) conceptual lead, the soil skeleton (solid phase) is considered in the present paper to be a very viscous non-Newtonian fluid. The general constitutive relationship defined for non-Newtonian material with nonlinear viscosity is given by:

\[
a^r = D \varepsilon
\]

where \( \varepsilon \) denotes the structural infinitesimal strain tensor which is a symmetric second order tensor that can be defined as \( Lu^s \) where \( L \) is a matrix for definition of strain and \( u^s \) is the displacement field of the solid phase. The dot represents a total derivative with respect to time. \( D \) is a fourth order tensor of viscosity for the constitutive law. For isotropic and isothermal material, \( D \) has only two independent parameters and can be expressed in the form with indices:

\[
D_{ijkl} = (\kappa - 2 \mu/3) \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})
\]

where \( \kappa \) and \( \mu \) are nonlinear viscous parameters. Li (1994) further defines, the bulk viscous parameter \( \kappa \) as a function of both the first strain invariant \( J_1 \) (=tre which denotes
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the trace of strain tensor \( \varepsilon \) and the second deviatoric strain invariant \( \mathcal{J}^D_2 (=0.5\varepsilon^D \varepsilon^D \) where \( \varepsilon^D \) is the deviatoric strain), and the shear viscous parameter \( \mu \) is expressed by the second deviatoric invariant \( \mathcal{J}^D_2 \) of strain in the forms:

\[
\kappa = \kappa_0 \exp[(1 + \alpha^D)(J_1 - J_{10})/A_1]
\]

\[
\mu = \mu_0 \exp[2(\sqrt{\mathcal{J}^D_2} - \sqrt{J_{10}^D})/A_2]
\]

where subscript "0" represents the initial state at time \( t = 0 \), \( A_i (i = 1,2) \) denotes the constitutive parameters that are assumed to be constants in this paper, \( \alpha^D \) is defined as the dilatant coefficient that is a function of \( \mathcal{J}^D_2 \) and \( \mathcal{J}_1 \). When \( i = j \) is assumed for a spherical stress-strain relationship only, then (19) and (21) reduce to Helm's (1995) earlier development.

It is important to discuss the term \( \nu \sigma' \) that plays such an important part in governing equation (9). From (19), one has the form:

\[
\nabla \sigma' = (\nabla D)\varepsilon + D(\nabla \varepsilon)
\]

(23)

Taking the derivative operator delta to (20) with assumptions (21 and (22) and substituting the result into the first term on the right-hand side of (23) give the expression as an alternative to equation (23):

\[
\nabla \sigma' = D \nabla \varepsilon + \dot{D} \nabla \varepsilon + R^\alpha
\]

(24)

where:

\[
R^\alpha = (\kappa \Delta \varepsilon t/A_1)(\triangledown \varepsilon \alpha^D - \alpha^D \triangledown \cdot \varepsilon t\varepsilon)\delta
\]

(25)

In a later section, equation (24) plays a key role in the governing equation for multiple processes.

**Complete mathematical description** In order to show a complete mathematical description, it is worthwhile to summarize all the fundamental equations involved in the derivation of governing equation (9).

**Effective stress**

\[
\sigma' = \sigma' + p \delta
\]

**Infinitesimal strain definition**

\[
\varepsilon = Lu^s
\]

**Constitutive law**

\[
\sigma' = D \varepsilon
\]

**Momentum conservation for a two-phase mixture**

\[
\nabla \sigma' + \rho_b \dot{b} = \rho_s a^s + \rho_w a^w
\]

**Momentum conservation for interstitial fluid**

\[
\rho_w a^w - n \nabla (p \delta) + n F^{\text{drag}} - \rho w b = 0
\]
Incompressibility condition
\[ \nabla \cdot q^b = \nabla \cdot [n v_w + (1-n) v^s] = 0 \]

Average drag force per volume
\[ F_{drag} = \{(\rho_w/2e) m^{-1} a' + [3\rho_w(\eta g / 2eK)^{0.5}] m^{-1} f(a') + \gamma_wK^{-1} q\} \]

There are a total of 28 equations and 28 unknowns \((u^s, u^w, \sigma', e, p, F_{drag})\), thereby constituting a complete mathematical description.

DISCUSSION AND COMPARISON TO PREVIOUS INVESTIGATIONS

Discussion of the mass conservation

If the following equations of state are assumed for water and individual solid grains:
\[ \rho_w = \rho_{w0}\exp(\beta_w p) \] (26)

and
\[ \rho_s = \rho_{s0}\beta_s \sigma_{v}s \] (27)

then (3) can be simplified to:
\[ \nabla \cdot q = -\nabla \cdot v^s - n\beta_w dp/dt - [(1-n)\rho_{s0}\beta_s / \rho_s]d\sigma_{v}s / dt \] (28)

where \(\beta_s\) and \(\beta_w\) are compressibility of an individual soil grain and water respectively and \(\sigma_{v}s\) is the total spherical stress on each grain. If one further assumes small strain for the individual solid grain and if \(\sigma_{v}s\) is decomposed into water pressure \(p\) and a resultant stress \(\sigma_{v}r's\), equation (28) becomes
\[ \nabla \cdot [n(v^w - v^s)] = -\nabla \cdot v^s - n\beta_w dp/dt - (1-n)(\beta_s d\sigma_{v}s / dt + \beta_s dp / dt) \] (29)

where the definition of \(q\) has been included in the left-hand side of (29).

Mass balance equation (29) is now contrasted to Zienkiewicz & Bettess’ (1982) equation which can be expressed:
\[ \nabla \cdot [n(v^w - v^s) + n(u^w - u^s)] = -\nabla \cdot v^s - n\beta_w dp/dt - (1-n)\beta_s dp/dt - \beta_s d\sigma_{v}' / dt \] (30)

where \(u^s\) and \(u^w\) represent the displacement fields respectively of the aquifer’s skeletal matrix and interstitial water, whereby \(v^s = \dot{u}^s\) and \(v^w = \dot{u}^w\). Whereas Zienkiewicz & Bettess’ effective stress term \(d\sigma_{v}' / dt\) in (30) can easily be perceived as representing our \((1 - n)d\sigma_{v}s / dt\) in (31), the second term \(\nabla \cdot d(u'n) / dt\) in the left-hand side of (30) can not be so easily explained. This term appears also in Zienkiewicz & Bettess’ expression of Darcy’s law, namely:
\[ n(v^w - v^s) + n(u^w - u^s) = -K \nabla h \] (31)

which contrasts to Gersevanov’s (1934), Biot’s (1941), Fukuo’s (1970), and Helm’s (1987) expressions of Darcy’s law. All others omit the term that includes the porosity.
time derivative. It appears this term is extraneous and should be omitted both from (30) and (31).

Fukuo (1970) employed a simplified form of (3) in which only the first term remains on the right-hand side. That is, he requires \( \nabla \cdot q = -\nabla \cdot \nu^s \) which actually is identical to the incompressibility condition \( \nabla \cdot q^b = 0 \).

**Discussion of the general momentum equation**

Assuming that the third term on the right-hand side of (24) is negligibly small when compared to other two terms and inserting (24) into the third term of (9) give an expression of soil displacement:

\[
c_1 \ddot{u}^s + c_2 \dot{u}^s + c_3 [D \nabla (L \dot{u}^s) + \dot{D} \nabla (Lu^s)] = R
\]

where two dots indicate the second derivative with respect to time \( (a^s = \ddot{u}^s) \). Vector \( R \) is defined by:

\[
R = c_4 b + c_5 \dot{q}^b + c_2 q^b + c_6
\]

Equation (32) is an informative governing equation describing solid matrix (aquifer) movement in multiple physical processes. The first term on the left-hand side is the inertial force, the second is the viscous force caused by viscous drag of pore water flow on the skeletal matrix, the third represents a viscous resistant force (friction) of the skeletal matrix on itself and the last is an apparent structural elastic force on the solid matrix. The third and the fourth terms together behave as a "Kelvin model" due to non-Newtonian behaviour. A Kelvin model has a spring and dashpot linked together in parallel. The first term and the fourth dominate wave propagation of the displacement field \( u^s \); the first and third dominate diffusion of the velocity field \( \nu^s \); the second and the fourth control diffusion of the displacement field \( u^s \). The right-hand vector \( R \) has several components which include changes in body force, drag force and other forces caused by the bulk flux \( q^b \) and \( \dot{q}^b \) whose divergence can be reduced by invoking incompressibility conditions.

It is interesting to compare equation (4) to the momentum equation given by Zienkiewicz & Bettess (1982):

\[
\nabla \sigma' + \rho^b b = \rho^b a^s + \rho^w \frac{d^2(nu')}{dt^2}
\]

where \( u' \) is the relative displacement. For the sake of the comparison, equation (4) is alternatively expressed by:

\[
\nabla \sigma' + \rho^b b = \rho^b a^s + \rho^w a'
\]

Comparing equation (34) to (35) shows that the difference is between the second terms on the right-hand side. One is written in terms of \( \frac{d^2(nu')}{dt^2} \), the other in terms of relative acceleration \( a' \). Recalling the definition \( a' = a^w - a^s \) and \( u' \) allows one to derive the following relation:

\[
a' \triangleq \ddot{u}' = \frac{d^2(nu')/dt - \ddot{n}u' - 2\ddot{n}u'}{n}
\]
Equation (34) can not equal (35) unless porosity $n$ is not a function of time. Alternatively, any porosity rates that occur in the final term in equation (34) are based on an error, similar to what was discussed in the previous section. It should be emphasized from a physical point of view, the validity of the final term on the right-hand side of (34) is questionable. The time derivatives of weighted average displacement, originally defined by Biot (1962), may not be physically meaningful for momentum conservation (34). In other words, equation (34) is an incorrect expression of momentum conservation.

Similarly, the fluid momentum equation given by Zienkiewicz & Bettess (1982) reflects an emphasis on porosity derivatives:

$$-\nabla p - \rho_w b = \gamma_w K^{-1} + \rho_w \frac{d^2(nu')/dt^2}{n} + \rho_w a^w$$  \hspace{1cm} (37)

Only when $a'$ equals $(d^2(nu')/dt^2)/n$ in (37) does the sum of the second and third terms on the right-hand side of (37) become $\rho_w a^w$ which is required in order to reach the following correct form:

$$-J = -\left[ \frac{a^w}{g} + \frac{\nabla p'}{\gamma_w \rho_w} + \frac{b}{\rho_w} \right] = -K^{-1}q$$  \hspace{1cm} (38)

Based on the above discussion of (34), equation (37) with derivatives of porosity with respect to time is a questionable description of momentum balance of pore fluid. Another important point is the following. According to (16)-(18), (38) is a simplified form of (16) and requires $a'=0$. Because (37) does not require $a'$ to be zero valued, there is a self-contradiction for (37) to reach (38). An editorial comment can also be made. The term $\gamma_w$ is missing in Zienkiewicz & Bettess’ (1982) equation.

In brief, the following points are important. Firstly, Darcy-Gersevanov’s law should not be introduced into the general momentum equation as if it is an independent formula as was done by Zienkiewicz & Bettess (1982). Darcy-Gersevanov’s law is a special case of momentum conservation of water flowing in porous material. Secondly, as a special case, Darcy-Gersevanov’s law requires all acceleration terms to be negligibly small when compared to velocity terms. Otherwise, application of this law into momentum balance considerations introduce self-contradictions. Thirdly, to be physically meaningful, derivatives are either material or local derivatives of physical properties. When a mixture is occupying a bulk volume, these derivatives can be weighted by the volume fraction and summed, for example $a^b$. These material derivatives and their volume fraction sums are substantially distinct in concept from simply taking the derivative of a physically meaningful vector sum, $\dot{q}^b$ vs. $a^b$. Unfortunately, the term $(d^2(nu')/dt^2)/n$, which represents time derivatives of the normalized relative displacement field $nu^s$ introduced by Biot (1962), are employed in (34) and (37) in place of physical meaningful relative forces per unit mass, $a'$.

Discussion of the momentum equation with small relative acceleration

If the relative acceleration $a'$ is very small when compared to phase inertial terms, (9) reduces to:
\[ \rho_s^* a^s - \gamma_w K^{-1} v^s - \nabla \sigma' - \rho_s^* b + \gamma_w K^{-1} q^b = 0 \]  
(39)

where \( \rho_s^* \) denotes the submerged density of a solid (or effective density). If one takes the divergence of each term in (39) with the assumptions of homogeneous, isotropic skeletal material, the bulk incompressibility condition (\( \nabla q^b = 0 \)), negligibly small change in body force (\( \nabla b = 0 \)) and negligibly small change of porosity in space (\( \nabla n = 0 \)), then one can rewrite (39) in terms of volume strain \( \varepsilon_v \) and effective stress without invoking a constitutive law:

\[ \rho_s^* \varepsilon_v^* - \gamma_w K^{-1} \varepsilon_v^* - \nabla \cdot \sigma' = 0 \]  
(40)

The coefficient for the first term in (40) is different from the one that appears in Fukuo’s (1970) equation, which is:

\[ \left( \rho_s^* + \frac{\rho_w}{e} \right) a^s - \gamma_w K^{-1} v^s - \nabla \cdot \sigma' - \rho_s^* b + \frac{\gamma_w K^{-1} q^b}{nG} = 0 \]  
(41)

in which phase densities (\( \rho_s \) and \( \rho_w \)) and hydraulic conductivity \( K \) are assumed to be constant. Relative acceleration and effective body force are also tacitly assumed to be negligibly small. Note that the signs of the first and second terms in (40) and (41) are different. The difference of signs is caused by different sign convention. The following discussion will centre on the coefficients that occur first terms. For convenience of comparison, one can rewrite (39) as:

\[ \left( \rho_s^* + \frac{\rho_w}{e} \right) a^s - \gamma_w K^{-1} v^s - \nabla \cdot \sigma' - \rho_s^* b + \frac{\gamma_w K^{-1} q^b}{nG} = 0 \]  
(42)

Taking the divergence of each term in (42) and making the same assumptions as Fukuo in (41), namely, that phase densities (or incompressibility condition), hydraulic conductivity and porosity are constant and that the effective body force is negligibly small, one has:

\[ \left( \rho_s^* + \frac{\rho_w}{e} \right) \varepsilon_v^* - \gamma_w K^{-1} \varepsilon_v^* - \nabla \cdot \sigma' - \frac{\rho_s^*}{nG} = 0 \]  
(43)

It is evident that the necessary condition for (43) to reach Fukuo’s form (41) requires the divergence of the last term in (43) to be zero, namely the term \( \nabla \cdot a^b = 0 \). It can be shown that Fukuo tacitly makes this assumption during his derivation. However, we shall demonstrate that this term can not be zero. In general, bulk acceleration \( a^b \) equals \( na^t + a^s \). The assumption of \( a^t = 0 \) in (41) requires therefore that \( a^b = a^s \). Taking the divergence of \( a^b \), one has the expression for the present case:

\[ \nabla \cdot a^b = \nabla \cdot a^s = \varepsilon_v \]  
(44)

Observing (41), one knows that \( \nabla a^b \) in (44) can not equal zero, otherwise the first term in (41) disappears and the dynamic description reduces to quasi-dynamic one inherently without an inertial term. A self-contradiction is imbedded in Fukuo’s governing equation for dynamic motion.
Discussion of the momentum equation with small absolute acceleration

Consider the case when slow movement of the solid frame $v^s$ only gradually approaches zero. Laminar pore water flow relative to the solid frame $q$ is not necessarily small and correspondingly viscous forces are assumed to dominate inertial forces. Acceleration terms on the left-hand side of (39) can be assumed to be much smaller than the velocity terms, $(K\rho_s^*/\gamma_w)v^s \ll v^s$. Also $a^w$ tacitly equals $a^s$ due to the previous assumption that relative acceleration $a^r$ is small compared to $a^s$. Then (39) reduces to:

$$v^s + \frac{1}{\gamma_w} K \nabla \sigma = q - \frac{1}{\gamma_w} K \rho_s^* b$$

(45)

or

$$q = \frac{1}{\gamma_w} K [\nabla \sigma + \rho_s^* b]$$

(46)

as an alternative expression of the Darcy-Gersevanov law. Furthermore, if one takes the divergence of each term of (46), one gets:

$$\dot{\varepsilon}_v = -\nabla \cdot [\gamma_w^{-1} K (\nabla \sigma + \rho_s^* b)]$$

(47)

Equation (47) becomes a diffusion equation in terms of volume strain if elastic skeletal material is assumed and $\nabla \rho_s^* b$ is small. The resulting poroelastic diffusion equation is essentially Biot’s (1941) and Mikasa’s (1965) equations of consolidation. These strain-based or Terzaghi’s (1925) pressure-based or Helm’s (1987) displacement-based diffusion equations can be used to model subsidence.

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