Inertial model of flow through porous media

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Abstract This paper demonstrates that pressure gradients which are observed in flow through porous media can be attributed to inertia forces due to changes in flow direction inherent to flow through porous media. Moreover, the paper shows that the magnitude of these pressure gradients, inertial in origin, are comparable with those empirically observed even in the case where dissipative forces are assumed to be negligible. It does this by presenting a mathematical model which includes higher-order derivatives of velocity which are usually neglected in equations of motion. These additional acceleration terms are analysed numerically on simple porous-media models. These models are constructed by placing, first, a single obstacle-sphere into a uniform flow and then by building a porous medium by adding to the number of spheres obstructing the flow. The pressure drop across the barrier of spheres is calculated numerically and compared with experimental findings.

INTRODUCTION

In flow through porous media the relation between an observed pressure drop and velocity is generally a nonlinear one. Darcy's flow region \((Re < 1)\) where an assumption of linearity is plausible, can be seen as the exception rather than the rule. Nonlinearity is expected due to the inertia forces which must occur in flow through a medium which imposes frequent changes in flow direction. However, the curvature and variable cross-section of flow channels do not enter into the porous media flow equations in their own right (Scheidegger, 1960). The acceleration of fluids passing through the curved channels is not rendered by the flow equations. Instead, the inertia forces are represented indirectly by means of correlations with viscous forces, capillary forces or the Reynolds number (Scheidegger, 1960; Bear, 1972; Liu et al., 1994).

Because inertia forces are related to first principles at one remove, there is a risk of the misinterpretation of the significance and identity of these forces (see, for example, the discussion by Firoozabadi et al., 1995).

The best we have been able to do is to represent inertia forces in flow through porous media indirectly. The fundamental equations of motion fail to account for these

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forces (Hassanizadeh & Gray, 1987). The reason for this becomes apparent when we consider inviscid flow through a porous medium. Irrespective of the curvature of the flow channels and the acceleration induced by the curvature, the equations of motion predict a zero pressure drop across any obstacle fully immersed in the ideal fluid. In this paper we attempt to show that the inertia forces which are in laminar flow owing to the curvature of the flow channels can be accounted for if the convective acceleration term, appearing in the equations of motion, is expanded by, at least, a second-order derivative of velocity.

FORMULATION OF THE PROBLEM

We consider a porous medium which is constituted by an array of spheres or cylinders, dispersed in an orderly manner such that we can assume that the flow through the medium is sinusoidal. This is shown in Fig. 1. We assume ideal fluid flow: incompressible and inviscid. We identify a characteristic length \( d \) in the porous medium over which we will seek to establish the pressure drop \( \Delta p \). In the case of an array of segments \( d \) which are identical and in alignment and which have a total length \( L \), the pressure drop across \( L \) becomes \( \Delta p(L/d) \). The characteristic length \( d \) is regarded as the minimum length over which sinusoidal flow can develop. We assume that \( d = \pi D \), where \( D \) is the diameter of the sphere. By defining our porous medium this way we are able to compare the results we obtain by means of numerical analysis, with those obtained experimentally. In the latter case these are obtained by measuring the terminal velocities of the steady-state fall of a single sphere in fluids under gravitational force. The drag coefficients (friction factors) \( f \) obtained experimentally can be compared with the pressure drop \( \Delta p \) on the basis of the equilibrium of forces acting on the fluid within \( d \) (see, for example, Bird et al., 1960):

\[
\frac{\Delta p}{d} = \frac{1}{d} \left( \frac{1}{2} \rho U^2 \right) f
\]

where \( U \) represents the steady-fall velocity or, equivalently, the average interstitial velocity of the porous medium, and \( \rho \) is the fluid density. The pressure drop represented by the coefficient \( f \) is affected by both viscous and inertia forces acting in the fluid. The analysis presented here cannot be verified against \( f \) in viscous flow regimes (Stokes's law regime and the intermediate regime). One might expect, however, that for Reynolds

![Sinusoidal flow](image-url)

**Fig. 1** Porous medium of spheres or cylinders.
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number \( Re = \frac{\rho UD}{\mu} \) \( 5 \times 10^2 < Re < 2 \times 10^5 \) where \( f \) remains almost constant and has values between 0.44 and 0.5, the ratio \( \Delta p/(\frac{1}{2}\rho U^2) \) should come close to measured values of \( f \), where \( \Delta p \) is the numerically obtained pressure drop. Some discrepancy between \( \Delta p/(\frac{1}{2}\rho U^2) \) and \( f \) is to be expected since a departure from symmetrical sinusoidal flow becomes clearly noticeable within this range of \( Re \) (see, for example, Batchelor, 1967).

For the sinusoidal flow to be physically meaningful, we assume that the sinusoid is the trajectory of the centre of mass (CM) of every section of the column of fluid approaching the sphere with velocity \( U \). The speed \( u \) of CM can then be expressed by the function \( u = \lambda U \sin x \) where the coefficient \( \lambda \) represents the elevation of the sinusoid with respect to the characteristic length \( d \). \( \lambda \) can be estimated on the basis of the mass conservation principle — a column of fluid approaching the sphere must also pass it at its peak. In the case of a sphere this leads to \( \lambda \approx 0.55 \). In the case of a cylinder \( \lambda = 1.0 \).

THE MATHEMATICAL MODEL

The mathematical model employed here is a one-dimensional modified Euler equation. A full description of the model is provided elsewhere (Pietlicki, 1995a,b). The approach admits inertia forces acting on the contact surfaces of fluid particles — understood as a material region of fluid — in addition to the pressure forces which are normally seen as the only forces with which the neighbourhood of the particles can act. The modified equation also admits a nonlinear distribution of velocity across the particles. The force \( F \) acting at the centre of mass of the particle and at points of contact with its neighbours on the left and right, is expressed by means of convection acceleration expanded by a second-order derivative of velocity:

\[
F = \frac{1}{2} \rho u \left( \frac{\delta u}{\delta x} + \frac{\delta^2 u}{\delta x^2} \right)
\]

(2)

where the coefficient \( \frac{1}{2} \) accounts for the fact that the same mass is engaged twice in the acceleration process — firstly when equation (2) is applied at the centre of mass of the particle and secondly when it is used at the contact points of the particle. The higher order term, rather than being treated as negligible, in fact plays a role in the solution process and in the boundary condition, in the same way as the viscosity term. In the latter case, the term can become small enough to lose all physical significance with respect to the value of forces occurring in the fluid in motion. At the same time the term retains significance in the mathematical representation of flow.

The equilibrium of all inertia and pressure forces gives the equation

\[
\frac{\delta p}{\delta x} + F_R - F_L + F_{CM} = 0
\]

(3)

from which pressure is obtained. In equation (3), \( F_{CM} \) represents the inertia force which acts at the centre of mass of the fluid particle. \( F_L \) and \( F_R \) represent inertia forces acting at the left and right contact points of the particle, respectively.

Using this approach, we determine the pressure distribution and pressure drop over the sinusoidal trajectory of fluid passing a sphere or a cylinder, where the speed of the
centre of mass of the "particle" entering the characteristic length \( d \) with velocity \( U \) is given by the pre-determined function \( u = \lambda U \sin x \).

**NUMERICAL ANALYSIS**

The pressure drop obtained on the basis of equation (3) is shown in Fig. 2: \( \Delta p/\left( \frac{1}{2} \rho U^2 \right) \) reaches a value of 0.8 at the end of the characteristic length \( d \). This result refers to flow around a cylinder (\( \lambda = 1.0 \)). The pressure drop we have calculated falls below observed values, since the drag coefficient for a cylinder varies in the region 0.9-1.2 for the Reynolds number considered here (see, for example, Shames, 1992). The result is closer to observed drag coefficients in the case of a sphere. For a sphere where \( \lambda = 0.55 \), the pressure drop \( \Delta p/\left( \frac{1}{2} \rho U^2 \right) \) is 0.44 which is the lower limit of the observed range of coefficients \( f \).

The pressure distribution which is shown in Fig. 2 does not represent the pressure distribution along the entire sinusoid of Fig. 1. For it to do so we need to set the characteristic length equal to \( 2\pi D \) rather than \( \pi D \) as has been assumed so far. In this, we make use of the symmetry of the sinusoid and the fact that \( \lambda \) acts just as a scaling factor. However, in the examples below we set the quantitative meaning of the results to one side, concentrating on a qualitative and two-way interpretation where integration is carried out over more than one \( \pi \) period. Namely, the results of integration over a \( 2\pi \) period can be regarded as a particular case of flow around two spheres (or cylinders) or alternatively, as a case of flow over a single sphere (cylinder) where the condition of fore-aft symmetry of the sinusoid, which was assumed in Fig. 1, can now be either maintained or relaxed. In this qualitative sense Fig. 3 can be interpreted in two ways either as showing the pressure drop for two identical spheres adjacent to each other, or the pressure distribution for a single sphere with a symmetrical speed function. Figure 4, on the other hand, can be interpreted as 2-sphere problem, where the spheres have
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The speed function is given by:

\[ u = U|\sin x| \]

Fig. 3 Flow across two characteristic lengths of identical curvature.

Fig. 4 Effect of variable curvature across two adjacent characteristic lengths.

different diameters (in relative proportion of 2:1). This can also be viewed as a single sphere case, with a shallower or steeper flow trajectory on one side of the sphere. Figure 4 demonstrates that the pressure drop across two spheres depends on the order of the spheres, if the diameters of these spheres are different. Figure 4 also illustrates the fact that where the pattern of flow becomes asymmetrical, as this occurs with the increase of \( Re \) the pressure distribution around a sphere (cylinder) becomes strongly nonlinearly dependent on the asymmetry. This kind of nonlinear, non-repetitive pattern of flow occurs, for example, at the entry of flow into a porous medium such as a packed bed of identical spheres (Dybbes & Edwards, 1984).
The asymmetry in the pressure distribution and the non-zero pressure drop shown in these examples is due to the presence of the second-order term in the mathematical model. The significance of this term is further illustrated in Fig. 3 and Fig. 5. Figure 3 shows that the successive refinement of the computational mesh stabilizes the solution relatively quickly, to the point where it becomes insensitive to further subdivision of the computational domain into smaller elements. However, refinement cannot be carried out indefinitely since machine accuracy is limited. This occurs much sooner than in the case of first-order derivatives. This loss of precision reduces the contribution of the second-order term, first locally and then globally. An example of this, using single precision numbers, is presented in Fig. 3. This shows convergence of solution at the division level of around 100 elements followed by solution divergence as the number of elements is increased to 500, 600 and 900. Figure 5 demonstrates the global loss of the contribution of the second-order term where the number of "spheres" considered is increased from 12 to 18 while maintaining the same level of refinement (320 elements). The 18-sphere solution coincides with the solution obtained on the basis of the Euler equation which is first-order accurate. These examples illustrate the character of the modified Euler equation. We do not expect that we can achieve convergence to an ultimately accurate solution by bringing the dimensions of "fluid particle" — interval $\delta x$, as close as possible to zero, and by concentrating exclusively on rendering first-order derivatives with sufficient accuracy. We expect to bring the reduction of these dimensions to a halt much earlier on so that the second-order terms are also representable within the machine accuracy available. The increased reliability of the model is a result of the balance between the smallness of the "fluid particle" and the complexity of its characterization.

![Diagram](image.png)

**Fig. 5** Vanishing second-order effect after exceeding machine accuracy in the 18-sphere case.
CONCLUSIONS

The results of the analysis presented in this paper indicate that inertia forces and the effects of these forces on pressure drop in porous media, on pore-size level, can be predicted on the basis of purely theoretical calculations. We have attempted to show that an estimation of drag coefficients is achievable on a theoretical basis. Hitherto, such estimation has only been possible, outside the creeping flow region, in the purely experimental domain.

The paper shows that the curvature of the flow channels of porous media can be accounted for on the basis of the fundamental laws of continuum mechanics. The inertia effects due to the curvature of channels are present in the laminar flow region at all velocities. There is no critical Reynolds number for the emergence of these effects. This confirms Scheidegger’s (1960) view which he arrived at through a critical analysis of hydraulic radius theory.

Further work is required to extend this analysis to include viscous forces and gravity. The inertia effects occurring on pore level need to be related to the parameters which characterize porous media on a macro-scale.

REFERENCES


