Interpretation of tracer tests in karst systems with unsteady flow conditions

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Abstract Tracer experiments in karst aquifers are frequently characterized by unsteady flow conditions. A numerical convective–dispersive model was developed for the evaluation of these experiments. The application of this model in comparison with a steady-state analytical solution shows the differences of the determined parameters.

Introduction

The interpretation of artificial tracer experiments in karst systems is usually performed by analytical solutions of the dispersion equation. In particular, the Multi-Dispersion-Model (MDM) of Maloszewski et al. (1992) has been successfully applied in karst systems (Maloszewski et al., 1994; Werner et al., 1997a,b). The MDM is a variation of the classical convection–dispersion model (Kreft & Zuber, 1978). The breakthrough curve of the tracer experiment is seen as the result of different flow paths. Step by step the transport parameters for the individual breakthrough curves are determined. However, the analytical solution requires a simplification of the regarded system like isotropy and homogeneity of the aquifer and a constant flow rate during the complete experiment. A constant flow rate is very unlikely, particularly in karst systems, because of the quick response of the aquifer to hydrologic changes. Zuber (1986) showed that it is not acceptable to use steady-state approaches for unsteady flow situations. Dzikowski (1995) used a convolution integral for the interpretation of artificial tracer tests with unsteady flow behaviour. In this paper a numerical model for the interpretation of such experiments is presented.
MODEL APPROACH

The numerical model was developed for the interpretation of tracer test in karst aquifers under variable flow conditions. It is a one-dimensional convective–dispersive model (Fig. 1) of ideal tracer transport under variable (in time) flow velocities. The solution to the transport equation (1) is found by applying finite element methods.

\[
\frac{\partial C}{\partial t} + v(t) \frac{\partial C}{\partial x} - D_L(t) \frac{\partial^2 C}{\partial x^2} = 0
\]

where
\[C\] = tracer concentration;
\[v(t)\] = flow velocity (as a function of time);
\[\alpha_L\] = longitudinal dispersivity;
\[D_L(t)\] = longitudinal dispersion; \[D_L = \alpha_L \cdot v(t)\] (as a function of time);
\[t\] = time variable;
\[x\] = space variable.

The following simplifications were assumed for the numerical model:
- the flow rate \[Q(t)\] and the flow velocity \[v(t)\] are the same in each element during one time step (the possible spatial variability of \[Q\] is not taken into account);
- the dispersivity is a constant parameter of the regarded system;
- the injection was done immediately (Dirac injection);
- the distance \[x\] between the injection and the detection point is fixed;
- the length \[L\] of one element is identical for all elements.

By applying (Istok, 1989) the weighted residual formulation (Galerkin method) of the transport, equation (1) can be re-written in the following form:

\[
[D(t)]\{C\} + [A(t)]\{0\} = 0
\]

with the vectors

\[
\{C\} = \begin{bmatrix}
C_1 \\
\vdots \\
C_k
\end{bmatrix}
\]

1, 2, 3... = node numbers
\[\text{Fig. 1 The concept of the numerical one-dimensional convection–dispersion model.}\]
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\[
\{ \frac{\partial C_i}{\partial t} \} = \left( \frac{\partial C_i}{\partial t} \right)
\]

where \( k \) = number of nodes and the matrices

\[
[D(t)] = \sum_{i=1}^{n}[D^{(i)}(t)]
\]

\[
[A(t)] = \sum_{i=1}^{n}[A^{(i)}(t)]
\]

where \( n \) = number of elements.

With a linear interpolation function the element matrices can be formulated (after Istok, 1989):

\[
[D^{(i)}(t)] = \int_{x_i}^{x_f} \left[ \frac{-1}{L} \right] \cdot D_L(t) \cdot \left[ \frac{-1}{L} \right] \, dx + \int_{x_i}^{x_f} \left[ \frac{x_j - x}{L} \right] \cdot v(t) \cdot \left[ \frac{-1}{L} \right] \, dx
\]

\[
\frac{D_L(t)}{L} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] + \frac{v(t)}{2} \left[ \begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right]
\]

\[
[A^{(i)}(t)] = \frac{L}{2} \left[ \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right]
\]

with \( L \) = length of the element; \( x_i, x_f \) = nodes of the element.

The solution of equation (2) in the time domain is obtained by applying the Crank-Nicholson formulation:

\[
(A(t) + \frac{1}{2} \Delta t[D(t + \Delta t)])\{C_{i + \Delta t}\} = (A(t) - \frac{1}{2} \Delta t[D(t)])\{C_i\}
\]

With specified initial values for \( \{C\}_{t=0} \), equation (8) can be solved for each time step.

The parameters

The unsteady flow conditions of a tracer test can be described by the discharge of a karst spring. This discharge is proportional to the flow rate in the karst system. It is assumed that the flow velocity \( v(t) \) used in equation (7) is dependent on the flow rate \( Q(t) \). The relationship of the parameter \( Q(t) \) to the parameter \( v(t) \) is given with the formulation of the theoretical cross-section area \( F_{med} \).

In the case of variable volume systems:

\[
V(t) = Q(t) \cdot t_b(t)
\]
\[ v(t) = \frac{Q(t)}{F_{med}} = \frac{Q(t)}{Q_{med}} \cdot \frac{x}{t_0} \]  
\[ F_{med} = \frac{Q_{med} \cdot t_0}{x} \]  

where

- \( V(t) \) = variable volume (as a function of time);
- \( v(t) \) = flow velocity (as a function of time);
- \( Q(t) \) = flow rate (as a function of time);
- \( Q_{med} \) = mean flow rate during the experiment;
- \( t_0 \) = mean transit time.

In the case of constant volume systems:

\[ Q(t) = \frac{V_0}{t_0(t)} \]  
\[ F = \frac{Q(t) \cdot t_0(t)}{x} = \frac{V_0}{x} = \text{constant} \]  
\[ v(t) = \frac{Q(t)}{F} = \frac{Q(t) \cdot x}{V_0} \]  

where

- \( F \) = cross-section area;
- \( V_0 \) = constant volume;
- \( t_0(t) \) = mean transit time (as a function of time).

With the equations (11 or 15) the flow velocity \( v(t) \) for each time step is specified:

The resulting dispersion \( D_L \) can be calculated by:

\[ D_L(t) = \alpha_L \cdot v(t) \]  

The numerical model depends on the two fit parameters mean transit time \( t_0 \) respectively the volume \( V \) and the dispersivity \( \alpha_L \). The value of the mean discharge is normally known.

**The initial condition**

As Dirac injection is difficult to implement as an initial condition in a numerical solution the following approximation is used to estimate the initial concentration in the second node in the case of Dirac impulse. With the analytical solution (17) after Kreft & Zuber (1978) the temporal concentration distribution of the second node (Fig. 1) of the numerical model can be calculated. These concentrations are taken as the initial condition for the model which is then shortened by the first element.
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\[ C(x,t) = \frac{M}{Q_{med} \cdot t_0} \sqrt{\frac{4\pi}{\alpha_L}} \frac{(t/t_0)^3}{(1 - (t/t_0)^2)} \exp \left[ -\frac{4\alpha_L}{x \cdot t_0} \right] \]  

(17)

where

\[ t_0 = \frac{V_0}{Q_{med}} \] (in case of constant volume)

\[ Q_{med} = \text{mean flow rate during the experiment;} \]

\[ M = \text{injected tracer mass.} \]

EXAMPLES

Three experiments performed in karstic systems were used to check how the variable

Fig. 2 Hydrogeological situation of the Danube-Aach system (western Swabian Alb) (modified after Hötzl, 1992).
A–B: profile section (see Fig. 3)
⊙ Location of the tracer injection in the first experiment
✷ Location of the tracer injection in the second experiment
Fig. 3 Geological section through the western Swabian Alb in the northeast (Hötzl, 1992).

Explanation of the symbols: Wg = Würm gravels, Wm = Würm moraine, ti1H = Upper Tithonian limestone, ti1Z = Lower Tithonian marl, ti1L = Lower Tithonian limestone, ki2-3 = Kimmeridge limestone, ki1 = Kimmeridge marl, ox2 = Oxfordian limestone, ox1 = Oxfordian marl, jd = Middle Jurassic marl and shale.

Fig. 4 Danube-Aach system: Tracer breakthrough curve and discharge of the sampling location Aachquelle. Above: first tracer experiment. Below: second tracer experiment. (The discharge data were provided by Landesanstalt für Umweltschutz (LfU) in Baden-Württemberg.)
flow model works. The variable flow volume of the karst systems is known from the analytical evaluation of several artificial tracer tests.

The first two tracer tests were performed in the Danube-Aach system, western Swabian Alb, Germany. The hydrogeological situation of this karstified area is shown in Fig. 2. The Danube River crosses this region from the west to the east. For about half of the year the river loses its water completely. About 12 km south of the main water loss the largest spring of Germany, the Aachquelle (mean discharge about

<table>
<thead>
<tr>
<th>Analytical Model (steady flow)</th>
<th>Numerical Model (steady flow) with analytical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean transit time [h]:</td>
<td>PEAK I 29.7  PEAK II 39.2</td>
</tr>
<tr>
<td>Dispersivity [m]:</td>
<td>PEAK I 31.3  PEAK II 49.2</td>
</tr>
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<tr>
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The Danube-Aach system: results from evaluating the first two peaks of the breakthrough curve in the first tracer experiment.

1 Analytical solution (steady state)
2 Numerical solution (steady state) with analytical determined values
3 Numerical solution (unsteady state) with analytical determined values
4 Numerical solution (unsteady state)
8.5 m³ s⁻¹), is located. The rocks of the Upper Jura (Malm) form the stratigraphic sequences, which fall gently towards the southeast (Fig. 3). The Malm strata is an interbedding of marlstone and karstified limestone. Due to the intensive fracturing of the region a hydraulic connection between the strata exists.

The first experiment (Fig. 4) was carried out with Uranine in the area of the main water loss (Fig. 2, location ©). This experiment was characterized by high water in the Danube River. This hydrologic event caused an increasing discharge of the Aachquelle spring where the sampling was performed. The analytical evaluation of the first two peaks of the resulting breakthrough curve with the multi-dispersion model (MDM) (Maloszewski et al., 1992) is shown in Fig. 5, ©. For comparability, the evaluation is limited to the first 50 hours of the experiment. However, the MDM is able to describe the whole breakthrough curve of the tracer test with five single peaks (Werner et al., 1997b).

As expected a calculation with the numerical model with the analytical parameters and without consideration of the unsteady flow conditions shows no great differences between the analytical and the numerical model (Fig. 5, ©). However, the unsteady-state calculation with the same parameters lead to considerable differences (Fig. 5, ®). A satisfactory fit is only possible with a modification of the fit parameters (Fig. 5, ®).

The transit times, which were determined under consideration of the unsteady-flow conditions, are lower than the steady flow transit times. The numerical model takes into account the higher velocities because of the increasing discharges.

The second tracer test (Fig. 4) was performed at another water loss of the Danube River (Fig. 2, location ©). The discharge of the Aachquelle spring was nearly constant during the first 100 h. The later part was characterized by a distinctive discharge peak. The total tracer (eosine) recovery was about 94%. The differences (Fig. 6) between the determined steady and unsteady flow transit times are only small, because of the constant discharges during the most time of the experiment. However, greater differences are only recognizable in the determined dispersivity values.

The data for the third example tracer test were taken from Rogalski (1988). The experiment was performed in the karst of the western Tatra Mountains (Poland) and is characterized by irregular discharge behaviour. Both the steady flow analytical and the unsteady flow numerical model can describe the breakthrough curve of the tracer test with two peaks. The analytical determined transit time (Fig. 7, ©) of the second peak is lower than the numerical value (Fig. 7, ®). The lower discharges at the beginning of the experiment were undervalued in the mean value (used for the unsteady analytical model) compared to the high discharges in the later part of the tracer test.

CONCLUSION

The calculations performed have shown that it is not valid to interpret tracer experiments under unsteady flow conditions with only a stationary model. The use of mean values for the discharge can be very critical if great differences between the individual discharge values exist. Table 1 shows an overview of the determined
Fig. 6 Danube-Aach system: results from evaluating the first two peaks of the breakthrough curve in the second tracer experiment.

1. Analytical solution (steady state)
2. Numerical solution (unsteady state)
Fig. 7 The interpretation of a tracer experiment in western Tatra Mts, Poland (after Rogalski, 1988).

<table>
<thead>
<tr>
<th></th>
<th>PEAK I</th>
<th>PEAK II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical Model</td>
<td>24</td>
<td>233</td>
</tr>
<tr>
<td>Mean transit time [h]:</td>
<td></td>
<td></td>
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<td>Dispersivity [m]:</td>
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<td>153</td>
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<tr>
<td>Numerical Model</td>
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<td>270</td>
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<tr>
<td>Mean transit time [h]:</td>
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<tr>
<td>Dispersivity [m]:</td>
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<td>290</td>
</tr>
</tbody>
</table>

① Analytical solution (steady state)
② Numerical solution (unsteady state)
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Table 1 Summary of the evaluation of three tracer experiments with steady and unsteady flow models.

<table>
<thead>
<tr>
<th>Discharge</th>
<th>Example</th>
<th>Mean transit time $t_0$</th>
<th>Dispersivity $\alpha_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continual increasing discharge</td>
<td><img src="image1.png" alt="Graph" /></td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Constant discharge</td>
<td><img src="image2.png" alt="Graph" /></td>
<td>=</td>
<td>+</td>
</tr>
<tr>
<td>Irregular course of the discharge</td>
<td><img src="image3.png" alt="Graph" /></td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

+ means that the unsteady flow value is higher than the steady flow value (mean value)
- means that the unsteady flow is lower than the steady flow value (mean value)

results. It is recognisable that the differences between the steady state and the unsteady state mean transit times depend on the course of the discharge. The dispersivity values for the three field examples are always higher for the unsteady flow regime. Consequently, the parameter differences must be considered if high accuracy is demanded. The numerical model developed is a useful tool to prove the influence of unsteady flow on the transport parameters.

REFERENCES


