Flow and transport in highly heterogeneous formations of multi-indicator permeability structure: an approximate analytical solution

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Abstract Flow of uniform mean velocity $U$ takes place in a heterogeneous medium made up from a matrix of conductivity $K_0$ and random inclusions of given shape and random conductivity $K$. Such a formation is coined as a "multi-indicator" and is representative of many sedimentary aquifers. The study derives simple, approximate, solutions for advective transport of passive solutes in such heterogeneous formations. Transport is characterized by the one particle trajectory statistical moments. The trajectory moments, and the related dispersion coefficients, as a function of time are determined by a few quadratures. The principal results obtained by the linear theory are recovered by the present analysis, which, however, is not limited by the low conductivity variability assumption. The asymptotic longitudinal dispersivity as a function of the logconductivity variance $\sigma_y^2$ is discussed, and asymptotic results for low- and highly-heterogeneous formations are presented. The setting time, i.e. the time for which the "Fickian" stage of transport is almost reached, is discussed. It is shown that the setting time for highly heterogeneous formations can be extremely large, and transport is characterized by a long "pre-Fickian" or "anomalous" stage.

Key words heterogeneous aquifer; porous media; solute transport

INTRODUCTION

Advective transport in heterogeneous porous formations has been the subject of several studies in recent years. In the past, the solution of the problem has been often studied with the aid of suitable space random functions that account for the spatially varying hydraulic properties. However, solution of the transport problem was hindered by its intrinsic nonlinearity, and a first-order analysis in the logconductivity variance $\sigma_y^2$ has been employed to solve the problem (e.g. Dagan, 1982; Gelhar & Axness, 1983; for a detailed analysis of the linear theories of transport, see Dagan, 1989). The first-order theories developed so far have been able to explain several features of the advective transport and predict the value of macrodispersivity for small to moderate values of $\sigma_y^2$. However, they have failed to capture other important mechanisms, e.g. tailing, non-Gaussianity of the trajectory distribution and other anomalies in transport.

The aim of the work is to study nonlinear transport of passive solutes, i.e. for any $\sigma_y^2$, for multi-indicator structures of permeability. The multi-indicator structure models the heterogeneous aquifer as a collection of inclusions of given shape and random conductivity $K$ implanted at random in a uniform matrix of conductivity $K_0$. Simple, approximate, solutions of advective transport of passive solutes in such
heterogeneous formations are derived. Solute transport is mainly ruled by the statistical distribution of the conductivity $K$, and the inclusions' volume fraction $n$, that is the relative volume of the domain occupied by the heterogeneous inclusions. Simple results are achieved by adopting circular (two-dimensional (2-D) formation) and spherical (3-D formation) shapes for the inclusions. The results are discussed and compared with those obtained in the past adopting the first-order theory. Advective transport is studied by means of the concentration spatial moments, which in turn are equal to the one particle trajectory statistical moments, for ergodic plumes. The trajectory moments, and the related dispersion coefficients, as a function of time, are determined by three quadratures.

The general framework of transport in multi-indicator formations is given in Dagan (2003), and hereafter we refer to that paper for any detail concerning the aquifer structure and the transport quantities. The semi-analytical approach outlined here is compared with the numerical simulations of Jankovic (2003).

**ASYMPTOTIC DISPERSIVITY**

The asymptotic, large-time longitudinal dispersivity $\alpha_L$ is evaluated through the second moment of longitudinal trajectory $X_{11}$, the relation between the two quantities being $\alpha_L = 1/(2U) \frac{dX_{11}}{dt}$, with $U$ the mean velocity. The second moment is given by

$$X_{11}(t) = \langle X_1^2(t; a) \rangle,$$

with $X_1' = X_1 - Ut$ the fluctuation around the mean of the displacement of a particle, injected at $x = a$ inside a domain where only one inclusion of given permeability contrast $\kappa$ is present. The solution for a cluster of inclusions is obtained by superimposing the solution for a random $\kappa$, and the effect of the surrounding inclusions is represented by an “effective” permeability $K_{eff}$ as background. The effective conductivity has been extensively studied in the past (for a review, see Dagan, 1989). With the above assumptions, the second moment $X_{11}(t)$ can be calculated with the following expression:

$$X_{11}(t) = \frac{n}{\omega(R)} \int_{\Omega} \{ X_1^2(t; a, \kappa, R) \} \, da \, dx \tag{1}$$

where $X_1'$ is the fluctuation of the trajectory of a particle injected at $x = a$ at $t = 0$ moving past a single inclusion of conductivity contrast $\kappa$ in a domain $\Omega$, $\omega(R)$ is the area/volume of the circular/spherical inclusion of radius $R$, $f(\kappa)$ is the pdf of $\kappa$, $n$ is the volume density of the inclusions.

The asymptotic dispersivity $\alpha_L(\infty)$ which is a measure of the spread of the plume in advective transport when the travel time is large enough to reach the “Fickian” stage, is obtained by derivation of (1) with respect to time and the subsequent limit of $t \to \infty$. Two simple asymptotic limits for $\alpha_L(\infty)$ are analysed first. When $\sigma_2^2 \ll 1$, i.e. for a low-heterogeneity aquifer, the dispersivity can be obtained by using an asymptotic expression for the dispersivity in a bimodal formation obtained by Eames & Bush (1999), arriving at the following result:

$$\alpha_L = \sigma_2^2 I_2 \tag{2}$$
where \( I_y = 8nR/(3\pi) \) (2-D), or \( I_y = 3nR/4 \) (3-D), is the logconductivity integral scale. The result (equation (2)) is identical with that obtained in the past by the first-order analysis of continuous random permeability fields (e.g. Dagan, 1992; Gelhar & Axness, 1993). The ability of the proposed model to capture the well-known first-order result strengthens the confidence in the model.

The opposite limit is for \( \sigma_r^2 \gg 1 \), i.e. for a highly heterogeneous formation. It can be obtained by realizing that, for large conductivity contrasts, advective transport is dominated by the low-conductivity blocks; an analytical expression for the dispersivity in a bimodal formation is given by Dagan & Lessoff (2001). By employing the latter, we arrive at the expression:

\[
\alpha_L = CI_y \exp(\sigma_r^2 / 2)
\]

with \( C = 1/2 \) (2-D), or \( C = 2/3K_{0Y}/K_{G} \) (3-D). Thus, the asymptotic longitudinal dispersivity for a highly heterogeneous formation increases exponentially with the logconductivity variance. This behaviour stems from the large travel time needed for the solute particles to pass through the low-conductivity inclusions, the number of which increases with the logconductivity variance. However, the limit (equation (3)) is of theoretical interest. In fact, the solute particles that move slowly inside the low-conductivity blocks escape laterally because of the effect of molecular diffusion. Hence, starting from a certain degree of heterogeneity, molecular diffusion begins to control advective transport and must be taken into account. This finding is consistent with previous numerical studies (e.g. Guswa & Freyberg, 2000; LaBolle & Fogg, 2001).

\[\text{Fig. 1} \quad \text{The asymptotic longitudinal dispersivity as function of the logconductivity variance } \sigma_r^2, \text{ for circular (2-D) and spherical (3-D) inclusions.}\]
Figure 1 represents the asymptotic longitudinal dispersivity as function of $\sigma_y^2$ for circular (2-D) and spherical (3-D) inclusions. The dispersivity is obtained by direct numerical integration of equation (1), with the trajectory fluctuation requiring an additional integration, which is carried out following the procedure of Lessoff & Dagan (2001). The first-order limit (equation (2)) is represented in Fig. 1 by a thick line. The large variance solution (equation (3)) is not represented in the figure, and it applies to very large values of $\sigma_y^2$. The striking result is that the first-order solution (equation (2)) is a good approximation of the asymptotic dispersivity up to $\sigma_y^2 = 4$ for the 2-D case, and up to $\sigma_y^2 = 2$ for the 3-D one. In fact, the averaging over the lognormal pdf balances the errors of the first-order analysis, that tends to overestimate dispersion for low permeability inclusions and to underestimate it for high permeability inclusions, resulting in an accurate result for the dispersivity. The balancing is less efficient for 3-D transport because of the bias induced by the effective conductivity, which is a function of $\sigma_y^2$.

TRANSIENT DISPERSIVITY

We now examine the time-dependent behaviour of the longitudinal dispersivity, which is quite important in defining the nature of advective transport. Similar to the previous Section, the starting point is equation (1). The small- and large time limits are obtained through (1) as follows:

$$
X_{11} = 2\sigma_{11}^2 t^2 ; \quad \alpha_t(t) = \sigma_{11}^2 t / U \quad (tU / \bar{U} \ll 1)
$$

$$
X_{11} = 2\alpha_{L}(\infty) Ut \quad (tU / \bar{U} \gg 1)
$$

(4)

The previous expressions represent the well-known Taylor regimes. Hence, at the earlier stages of the transport process the dispersivity grows linear with time, proportional to the velocity variance. Conversely, at large time the dispersivity tends to the constant limit $\alpha_{L}(\infty)$, that was the subject of the previous section, and is a function of the logconductivity variance.

To demonstrate the results, in Fig. 2 we depict the dimensionless dispersivity $\alpha_c(t)$ as a function of dimensionless time for a few values of the logconductivity variance $\sigma_y^2$, for both circular (2-D) and spherical (3-D) inclusions. The results are obtained by numerical integration of equation (1). The small and large time limits (equation (4)) are clearly visible in Fig. 2. Both limits depend on the velocity variance and on the asymptotic, large-time dispersivity, that were derived in the previous Section. Our interest now is the transient behaviour, and in particular the setting time, $t_F$, that is defined as the time needed to reach the asymptotic dispersivity. The setting time is a measure of the duration of the pre-Fickian stage of transport. Starting with the first-order approximation, the setting time does not depend on the logconductivity variance; it is a function of the Eulerian velocity autocorrelation function only, which in turns depends on the particular structure of the random conductivity field (see e.g. Dagan, 1989). For the typical negative-exponential or Gaussian covariance structures of $Y$, the setting time is usually $t_F = O(10^4)$. The latter result is recovered in the present analysis, as shown by the curve for $\sigma_y^2 = 0.1$ of Fig. 2. Thus, consistent with the first-order solution, the setting time does not depend on the logconductivity variance when $\sigma_y^2 < 1$. 

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Things change significantly for higher values of the logconductivity variance, the setting time depending strongly on \( \sigma_f^2 \); in particular, a systematic increase of the setting time is observed. It is seen in Fig. 2 that for \( \sigma_f^2 = 4 \), \( t_F = O(10^3) \), two orders of magnitude larger than before. The increase of the setting time is caused by the large travel time that part of the solute body experiences when crossing a low-conductivity zone. The Fickian limit is reached when the solute travelling through the porous media has sampled the spatially varying velocity field. Such sampling is limited by the slow movement of the solute inside the low-conductivity blocks of the aquifer. Hence, more time is needed for the solute to sample the heterogeneous system, leading to an increase of the setting time. The consequence is that transport in highly heterogeneous aquifers, i.e. when \( \sigma_f^2 \) is larger than unity, can be characterized by a long, non-Fickian or anomalous regime that may last for a considerable period. These features, which are sometimes observed in the field experiments, are not captured by the first-order theories of transport.

**SUMMARY AND CONCLUSIONS**

The longitudinal dispersivity of a passive solute in a multi-indicator formation was studied. The transient longitudinal dispersivity as a function of the logconductivity variance and the density ratio \( n \), was analysed. Following the well-known pattern, the dispersivity grows linearly with time at the earlier stages of transport, and tends to a
constant asymptotic limit at large time. The asymptotic limit grows with $\sigma_y^2$; in particular, its asymptotic limit for small variance is identical to the value obtained by the first-order approximation. The striking result is that the first-order approximation seems to hold for values of the logconductivity variance much larger than expected, up to $\sigma_y^2 = 4$. This results from a compensation of errors associated with the first-order approximation. In contrast, for $\sigma_y^2 >> 1$, the asymptotic dispersivity grows exponentially with $\sigma_y^2$. This behaviour is caused by the hold-up of the solute by low-conductivity inclusions. However, such a large increase of the dispersivity is generally limited by molecular diffusion.

Of particular interest is the travel time needed to reach the asymptotic dispersivity, denoted as setting time $t_F$. It delimits the period in which transport is non-Fickian, or "anomalous". For weak heterogeneity, the setting time is usually $t_F = O(1)$; the latter result is in agreement with the first-order analysis of transport. However, when the logconductivity variance grows, the setting time increases dramatically, orders of magnitude larger than the one predicted by the first order analysis. Such an increase of $t_F$ is caused again by the large travel time needed for the solute particles to cross the low conductivity blocks. The consequence is that transport in highly heterogeneous formations can be characterized by a very prolonged, non-Fickian stage, with dispersivity growing continuously with time.

REFERENCES


